The classification of the cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–modules

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Abstract

In this paper we classify all the cyclic finite dimensional indecomposable modules of the perfect Lie algebras $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$, given by the semidirect sum of the simple Lie algebra A_n with its standard representation. Furthermore, using the embeddings of the Lie algebras $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ in $\mathfrak{sl}(n+2)$, we show that any finite dimensional irreducible module of $\mathfrak{sl}(n+2)$ restricted to $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ is a cyclic module and that any cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–modules can be constructed as quotient module of the restriction to $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ of some finite dimensional irreducible $\mathfrak{sl}(n+2)$–modules. This explicit realization of the cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–modules plays a role in their classification.

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1 Introduction

Surely the construction and the (even partial) classification of the indecomposable modules of a given Lie algebra is a very hard and (in most of the cases) an almost unsolved problem. In fact only for the class of all semisimple Lie algebras such classification has been accomplished, while, for the non semisimple Lie algebras very few results have been, until now, achieved.

In the case of semisimple Lie algebra the classification of all indecomposable module can be summarized by saying that any indecomposable module is an irreducible modules and for a given simple Lie algebra $\mathfrak{s}$ the set of irreducible modules is classified by an element in $\mathbb{N}^n$, where $\mathbb{N}$ is the set of the natural number and n is the rank of $\mathfrak{s}$. In fact this element of $\mathbb{N}^n$ is a dominant weight of $\mathfrak{g}$ see [13], [12] or the next section for more details.

In order to obtain similar result for non semisimple Lie algebra one has to identify a distinguished class of indecomposable representations for which one could expect to obtain a reasonable classification. This seems to be possible in two ways.

First, one can consider the embeddings of a given Lie algebra $\mathfrak{g}$ into a semisimple Lie algebra $\mathfrak{s}$ (or other Lie algebras whose indecomposable representations are at least partial known, see for instance [1] for an example where a not semisimple Lie algebra, in fact a truncated current Lie algebra [5] is used) and to use the well known irreducible finite dimensional modules of $\mathfrak{s}$ to study and classify the indecomposable $\mathfrak{g}$–modules obtained by restriction. Such approach to the theory of the indecomposable module of non semisimple Lie algebra has been pioneered by Douglas and Premat in [8] and in the PhD Thesis of Douglas and further developed by Deguise, Douglas, Premat, Repka in [6],[7],[17],[9] and by Minniti, Salari and the author in [1].

Second, one can select a particular class of indecomposable modules which satisfy suitable properties like to be cyclic or uniserial. In this contest in fact Cagliero and Szechtman [3] in a recent beautiful paper have classified all uniserial $\mathfrak{g}$–modules of the perfect Lie algebras $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$, where $V(m)$ is the irreducible $\mathfrak{sl}(2)$–module with highest weight $m \geq 1$, Previously only the case corresponding to the Lie algebra $\mathfrak{sl}(2) \ltimes V(1)$, [16] (where actually all the indecomposable representations with one generator had been classified), was known.

The aim of this paper is to proceed further into these lines of though, using both approaches. We shall indeed consider the perfect Lie algebras $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ given by the semidirect sum of the simple Lie algebra A_n with its standard representation.

The main results is a complete classification of all cyclic finite dimensional $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module with $n \geq 1$ (observe that the case $n = 1$ was already obtained by Piard in [16]), i.e., the indecomposable modules with only one generators, and therefore as subcase all the uniserial $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–modules, see Theorem 4.7. We shall namely show that such indecomposable modules are classified by particular bounded subsets of integer numbers see Definition 4.5.

In order to achieve such result we shall make use of the fact that the perfect Lie algebra $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ can be embedded in the simple Lie algebra $\mathfrak{sl}(n+2)$ see [8].

More precisely we first show how to associate a set $\mathcal{J}(V)$ of integer numbers to any cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module V and that this correspondence is injective: to inequivalent cyclic modules correspond different subsets of integer numbers see Theorems 3.25 and 3.27. Then we use the restriction to $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ of the finite dimensional irreducible $\mathfrak{sl}(n+2)$–modules in order to construct for any such set of integer numbers a corresponding cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module see Theorem 4.6. Here it is interesting to note that any $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ module obtained by restriction of a finite dimensional irreducible $\mathfrak{sl}(n+2)$–modules V on $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ is itself a cyclic module generated by the highest weight vector of V.

The paper is organized as follows. In Section 2 we collect all the needed properties of the perfect Lie algebras $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ and of their embeddings in $\mathfrak{sl}(n+2)$. Furthermore we brifely describe the finite irreducible module of the simple Lie algebra of type A_n together with the basis of such modules recently founded by Feigin, Fourier and Littelmann [11], which will play a important role in the last section. In Sections 3 we first present the definition of cyclic module together with some general result on cyclic module of perfect Lie algebras. Then we obtain a detailed description of the finite dimensional cyclic modules of the perfect Lie algebra $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ as direct sum of irreducible $\mathfrak{sl}(n+1)$–modules and determine how the radical of $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ intertwine them. Finally we show how any finite dimensional cyclic module V is determined by a bounded subset of integer numbers $\mathcal{J}(V)$. In Section 4 we study the indecomposable modules of $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ which arise a restriction to them of the finite irreducible $\mathfrak{sl}(n+2)$–modules, showing that they are cyclic and that any cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module can be viewed as their quotient. This latter result together with those obtained in section 3 complete the classification of the $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–modules.

2 The perfect Lie algebra $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$

In this section we will describe into the details needed for our purposes the perfect Lie algebra $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ given by the semidirect sum of the classical simple Lie algebra $\mathfrak{sl}(n+1)$ with its standard representation [10].
Throughout this paper, all the Lie algebras and their modules (which are supposed to be always finite dimensional) are over the ground field $\mathbb{C}$ of the complex numbers.

Let $\mathfrak{s}$ be a simple complex Lie algebra, $\mathfrak{h} \subset \mathfrak{s}$ a Cartan subalgebra, $\mathfrak{h}^*$ its complex dual and $\Delta = \Delta(\mathfrak{s}, \mathfrak{h}) \subset \mathfrak{h}^*$ the set of roots of $\mathfrak{h}$ in $\mathfrak{s}$. Denote by $(\ ,\ )$ the Killing form on $\mathfrak{g}$. The induced form on $\mathfrak{h}^*$ will be denoted by $(\ ,\ )$ as well.

For $\alpha \in \Delta$, let $H_\alpha \in \mathfrak{h}$ be the corresponding coroot and let $\mathfrak{s}_\alpha = \{X \in \mathfrak{s} \mid [H, X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}$ its root–space.

A set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \Delta$ is a basis of $\mathfrak{h}^*$ such that any root α belongs either to the monoid $\mathbb{Z}_{\geq 0}\Pi$ or to its opposite. Accordingly we can view Δ as the disjoint union of positive roots $\Delta^+_\Pi = \mathbb{Z}_{\geq 0}\Pi \cap \Delta$ and negative roots $\Delta^-_\Pi = -\Delta^+_\Pi$. (Since we are going to choose a set of simple roots once for ever we shall simply write $\Delta^\pm$ instead of $\Delta^\pm_\Pi$). This

decomposition of Δ induces the Cartan decomposition of $\mathfrak{s}$:

$$\mathfrak{s} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$$

where $\mathfrak{n}^+ = \oplus_{\alpha\in\Delta^+}\mathfrak{s}_\alpha$ and $\mathfrak{n}^- = \oplus_{\alpha\in\Delta^-}\mathfrak{s}_\alpha$ are maximal nilpotent subalgebras.
Let E_α be a basis of $\mathfrak{s}_\alpha$ and let F_α in $\mathfrak{s}_{-\alpha}$ be defined by the requirement $[E_\alpha, F_\alpha] = H_\alpha$.
Denote by $\mathcal{U}(\mathfrak{s})$, $\mathcal{U}(\mathfrak{n}^+)$, $\mathcal{U}(\mathfrak{n}^-)$ the universal enveloping algebras of $\mathfrak{s}$, $\mathfrak{n}^+$, $\mathfrak{n}^-$ respectively. (More in general $\mathcal{U}(\mathfrak{g})$ will denote the universal enveloping algebra of the Lie algebra $\mathfrak{g}$.)

If the simple Lie algebra $\mathfrak{s}$ is the complex Lie algebra $\mathfrak{sl}(n+1)$, we can choose as Cartan subalgebra the set of all diagonal matrices in $\mathfrak{sl}(n+1)$, then the Cartan decomposition becomes the usual triangular decomposition of $\mathfrak{sl}(n+1)$ in into the direct sum of strictly upper triangular, diagonal, and strictly lower triangular matrices.

If H is the diagonal matrix with $h_1, \dots, h_{n+1}$ on the main diagonal we set $\epsilon_j(H) = h_j$. Then $\Delta(\mathfrak{g}, \mathfrak{h}) = \{\epsilon_i - \epsilon_j, \;\; i, j = 1, \dots n+1, \quad i \neq j\}$ and we may take as set of simple roots $\Pi = \{\, \alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2 - \epsilon_3, \dots,\ \alpha_n = \epsilon_n - \epsilon_{n+1}\}$, hence the roots of $\mathfrak{sl}(n+1, \mathbb{C})$ are of the form $\pm(\alpha_p + \alpha_{p+1} + \cdots + \alpha_q)$ for some $1 \leq p \leq q \leq n$. Following [11] we set for $1 \leq p < q \leq n$:

$$\alpha_{p,q} = \alpha_p + \dots \alpha_q \qquad H_{p,q} = H_{\alpha_{p,q}} \qquad E_{p,q} = E_{\alpha_{p,q}} \qquad F_{p,q} = F_{\alpha_{p,q}},$$

and for our convenience also $\alpha_{i,i} = \alpha_i$, $H_{i,i} = H_{\alpha_i} = H_i$, $E_{i,i} = E_{\alpha_i} = E_i$, and $F_{i,i} = F_{\alpha_i} = F_i$.

Let $A_{i,j}$ be the $(n+1)\times(n+1)$ matrix with 1 in the i, j position and 0's everywhere else. Then the coroot $H_{p,q}$ is the matrix $H_{p,q} = A_{p,p} - A_{q,q}$, and the elements $E_{p,q}$ and $F_{p,q}$ are respectively $E_{p,q} = A_{p,q+1}$ and $F_{p,q} = A_{q+1,p}$, $1 \leq p \leq q \leq n$.

An element of $\mathfrak{h}^*$ is called a weight. The set $P = \{\lambda \in \mathfrak{h}^* |\ \lambda(H_\alpha) \in \mathbb{Z},\ \forall \alpha \in \Delta\}$ is said the set of integral weights of $\mathfrak{s}$. A weight λ of P is said dominant if $\lambda(H_\alpha) \geq 0$ for any simple root α, let us denote by Λ the subset of the integral dominant weights.

A basis of $\mathfrak{h}^*$ is given by the fundamental weights $\{\omega_1, \dots, \omega_n\}$ defined by the relations $\omega_i(H_j) = \delta_{i.j}$ for $i, j = 1 \dots n$, where $\delta_{i.j}$ is the usual Kronecker delta. With respect to this basis the integral dominant weights in $\mathfrak{h}^*$ can be written as $\lambda = \sum_{i=1}^n \lambda_i \omega_i$, $\lambda_i \in \mathbb{N}$. It is also convenient to set $\omega_{n+1} = \omega_0 = 0$.

The complex finite dimensional irreducible representations of $\mathfrak{sl}(n+1)$ are parametrized by the dominant integral weights. We denote by $V(\lambda)$ the finite dimensional irreducible $\mathfrak{sl}(n+1)$–module corresponding to the integral dominant weight λ, and by v_λ its highest weight vector i.e., a non trivial vector in $V(\lambda)$ of weight λ annihilated by $\mathfrak{n}^+$: $Hv_\lambda = \lambda(H)v_\lambda$ for all $H \in \mathfrak{h}$ and $\mathfrak{n}^+ v_\lambda = 0$, which generates $V(\lambda)$: $V(\lambda) = \mathcal{U}(\mathfrak{s})v_\lambda$.

An element μ of $\mathfrak{h}^*$ is said a weight of an irreducible finite dimensional module $V(\lambda)$ if the weight space $V_\mu = \{v \in V(\lambda)|\ Hv = \mu(H)v\ \forall H \in \mathfrak{h}\}$ is different from the zero vector. Denote by $P(\lambda)$ the set of all weights of $V(\lambda)$. The module $V(\lambda)$ may be decomposed as the direct sum of its weight spaces:

$$V(\lambda) = \bigoplus_{\mu\in P(\lambda)} V_\mu. \tag{2.1}$$

In what follows we shall need the explicit basis of $V(\lambda)$ (for the simple Lie algebra of type A.) constructed in a beautiful paper of Feigin, Fourier and Littelmann [11]. This basis conjectured by Vinberg is related to the notion of Dyck path.

Definition 2.1 *[11]. A Dyck path is a sequence*

$$\mathbf{p} = (\beta(0), \beta(1), \ldots, \beta(k)), \qquad k \geq 0;$$

of positive roots satisfying the following conditions:

1. *If $k = 0$, then $\mathbf{p}$ is of the form $\mathbf{p} = (\alpha_i)$ for some simple root α_i.*
2. *If $k \geq 1$, then:*
   (a) *the first and last elements are simple roots. More precisely, $\beta(0) = \alpha_i$ and $\beta_k = \alpha_j$ for some $1 \leq i < j \leq n$.*
   (b) *the elements in between obey the following recursion rule. If $\beta(s) = \alpha_{p,q}$ then the next element in the sequence is of the form either $\beta(s+1) = \alpha_{p,q+1}$ or $\beta(s+1) = \alpha_{p+1,q}$.*

Let $\mathcal{S}(\mathfrak{n}^-)$ denote the symmetric algebra of $\mathfrak{n}^-$. Then for a multi–exponent $\mathbf{s} = (s_\beta)_{\beta\in\Delta^+}$, $s_\beta \in \mathbb{Z}_{\geq 0}$, let $F^{\mathbf{s}}$ be the element

$$F^{\mathbf{s}} = \prod_{\beta\in\Delta^+} F_\beta^{s_\beta} \in \mathcal{S}(\mathfrak{n}^-).$$

Theorem 2.2 *[11] Let $\lambda = \sum_{i=1}^n \lambda_i\omega_i$ be an integral dominant $\mathfrak{sl}(n+1)$–weight and let $S(\lambda)$ be the set of all multi-exponents $\mathbf{s} = (s_\beta)_{\beta\in\Delta^+} \in \mathbb{Z}_{\geq 0}^{\Delta^+}$ such that, for all Dyck paths $\mathbf{p} = (\beta(0), \ldots \beta(k))$:*

$$s_{\beta(0)} + s_{\beta(1)} + \cdots + s_{\beta(k)} \leq \lambda_i + \lambda_{i+1} + \ldots \lambda_j \tag{2.2}$$

where $\beta(0) = \alpha_i$ and $\beta(k) = \alpha_j$. Then if v_λ is a highest vector of $V(\lambda)$, the set $F^{\mathbf{s}}v_\lambda$ with $\mathbf{s} \in S(\lambda)$ forms a basis of $V(\lambda)$, which we shall call the Feigin Frenkel Littelmann basis ((FFL) basis) of $V(\lambda)$.

The standard module of $\mathfrak{sl}(n+1)$ on $\mathbb{C}^{n+1}$ coincides with the highest weight module $V(\omega_1)$. A suitable basis for this space together with the action on it of the elements of $\mathfrak{sl}(n+1)$ is given by

Proposition 2.3 *Let u_{ω_1} be a highest weight vector of $V(\omega_1)$ then*
1) the set:

$$\mathcal{S} = \{v_{\omega_1}, F_1 v_{\omega_1}, \cdots, F_{1i}v_{\omega_1}, \cdots, F_{1n}v_{\omega_1}\} \tag{2.3}$$

is a basis of $V(\omega_1)$.
2) The action of $\mathfrak{sl}(n+1)$ on $\mathcal{S}$ is given by the relations

$$\begin{aligned}
&H_j v_{\omega_1} = \delta_{j1} v_{\omega_1} && H_j F_{1,i} v_{\omega_1} = \delta_{j,i+1} F_{1,i} v_{\omega_1} - \delta_{j,i} F_{1,i} v_{\omega_1} \\
&E_{j-h,j} v_{\omega_1} = 0 && E_{j-h,j} F_{1,i} v_{\omega_1} = \delta_{j,i} F_{1,j-h-1} v_{\omega_1}, \quad F_{1,0} = 1 \\
&F_{j-h,j} v_{\omega_1} = \delta_{1,j-h} F_{1,j} v_{\omega_1} && F_{j-h,j} F_{1,i} v_{\omega_1} = \delta_{j-1,i+h} F_{1,j} v_{\omega_1} \quad i, j = 1, \ldots, n \;\; 0 \leq h < j.
\end{aligned} \tag{2.4}$$

Proof 1) That the set $\mathcal{S}$ (2.3) is a basis of the $\mathfrak{sl}(n+1)$–module $V(\omega_1)$ can be checked using Theorem 2.2 [11]. However, in this very simple case, it can be also shown directly. Indeed since for any $1 \leq i \leq n$ the subalgebra spanned by the elements $\{H_{1i}, E_{1i}, F_{1i}\}$ is isomorphic to the simple Lie algebra $\mathfrak{sl}(2)$ and since $H_{1i}v_{\omega_1} = \omega_1(\alpha_{1i})v_{\omega_1} = v_{\omega_1}$, from the theory of the representations of $\mathfrak{sl}(2)$ follows that $F_{1,i}v_{\omega_1} \neq 0$. Further, since the element $F_{1,i}v_{\omega_i}$ has weight $\omega_i - \alpha_{1,i}$, two different elements of the set $\mathcal{S}$ belong to two different weight–spaces. To show that the set $\mathcal{S}$ is a basis of $V(\omega_1)$ it suffices therefore to show that its number of elements is equals the dimension $n+1$ of $V(\omega_1)$, which is obvious.
2) The equations (2.4) follow immediately from the commutation relations of $\mathfrak{sl}(n+1)$, which for the convenience of the reader are explicitly given in the next proposition. (see equation (2.5)).

□

Therefore setting $P_1 = v_{\omega_1}$, $P_{j+1} = F_{1,j}v_{\omega_i}$, $1 \leq j \leq n$ and using the basis of $\mathfrak{sl}(n+1)$ given above we have

Proposition 2.4 *The perfect Lie algebra $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ is the Lie algebra spanned by the elements*

$$\left\{H_i, E_{p,q}, F_{p,q}, P_j\right\} \qquad 1 \leq i \leq n, \quad 1 \leq p \leq q \leq n, \quad 1 \leq j \leq n+1$$

whose non trivial Lie brackets are:

$$\begin{aligned}
&\left[H_i, E_{p,q}\right] = (\alpha_i, \alpha_{p,q})E_{p,q} && \left[H_i, F_{p,q}\right] = -(\alpha_i, \alpha_{p,q})F_{p,q} \\
&\left[E_{p,q}, E_{r,s}\right] = \delta_{q,r-1}E_{p,s} - \delta_{p,s+1}E_{r,q} && \left[F_{p,q}, F_{r,s}\right] = -\delta_{q,r-1}F_{p,s} + \delta_{p,s+1}F_{r,q} \\
&\left[E_{p,q}, F_{p,q}\right] = H_{p,q} && \\
&\left[E_{p,q}, F_{p,s}\right] = -F_{q+1,s} \quad \textit{if } s > q && \left[E_{p,q}, F_{p,s}\right] = -E_{s+1,q} \quad \textit{if } q > s \\
&\left[E_{p,q}, F_{r,q}\right] = E_{p,r-1} \quad \textit{if } p < r && \left[E_{p,q}, F_{r,q}\right] = F_{r,p-1} \quad \textit{if } p > r \\
&\left[H_i, P_j\right] = \delta_{ij}P_j - \delta_{i,j-1}P_j && \\
&\left[E_{p,q}, P_j\right] = \delta_{q,j-1}P_p && \left[F_{p,q}, P_j\right] = \delta_{p,j}P_{q+1}.
\end{aligned} \tag{2.5}$$

In [10] Douglas and Repka have classified all the embeddings of the perfect but not simple Lie algebra $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ in the simple Lie algebra $\mathfrak{sl}(n+2)$. Using for the Lie algebra $\mathfrak{sl}(n+2)$ the same notion of above but with the indices running from 1 to $n+1$ their results may summarized as

Theorem 2.5 *[10] There are, up to inner automorphism, two inequivalent embeddings Φ, Θ of*

$\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ *in the simple Lie algebra* $\mathfrak{sl}(n+2)$*:*

$$
\begin{aligned}
&\Phi : \mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1} \longrightarrow \mathfrak{sl}(n+2)\\
&\Phi(H_i) = A_{i+1,i+1} - A_{i+2,i+2} \ \textit{(the element } H_{i+1} \textit{ of } \mathfrak{sl}(n+2)\textit{)} \quad 1 \le i \le n\\
&\Phi(E_{p,q}) = A_{p+1,q+2} \ \textit{(the element } E_{p+1,q+1} \textit{ of } \mathfrak{sl}(n+2)\textit{)} \quad 1 \le p \le q \le n\\
&\Phi(F_{p,q}) = A_{q+2,p+1} \ \textit{(the element } F_{p+1,q+1} \textit{ of } \mathfrak{sl}(n+2)\textit{)} \quad 1 \le p \le q \le n\\
&\Phi(P_j) = A_{j+1,1} \ \textit{(the element } F_{1,j} \textit{ of } \mathfrak{sl}(n+2)\textit{)} \quad 1 \le j \le n+1
\end{aligned}
\tag{2.6}
$$

and

$$
\begin{aligned}
&\Theta : \mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1} \longrightarrow \mathfrak{sl}(n+2)\\
&\Theta(H_i) = -A_{i+1,i+1} + A_{i+2,i+2} \ \textit{(the element } -H_{i+1} \textit{ of } \mathfrak{sl}(n+2)\textit{)} \quad 1 \le i \le n\\
&\Theta(E_{p,q}) = -A_{q+2,p+1} \ \textit{(the element } -F_{q+1,p+1} \textit{ of } \mathfrak{sl}(n+2)\textit{)} \quad 1 \le p \le q \le n\\
&\Theta(F_{p,q}) = -A_{p+1,q+2} \ \textit{(the element } -E_{p+1,q+1} \textit{ of } \mathfrak{sl}(n+2)\textit{)} \quad 1 \le p \le q \le n\\
&\Theta(P_j) = A_{1,j+1} \ \textit{(the element } E_{1,j} \textit{ of } \mathfrak{sl}(n+2)\textit{)} \quad 1 \le j \le n+1.
\end{aligned}
\tag{2.7}
$$

Remark 2.6 *Both embeddings of* $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ *into* $\mathfrak{sl}(n+2)$, Φ, *and* Θ *define on the space* $\mathbb{C}^{n+2}$ *a structure of* $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$*–module which we will denote respectively* $\mathbb{C}^{n+2}_{\Phi}$ *and* $\mathbb{C}^{n+2}_{\Theta}$*. Both modules are true and indecomposable. The Jordan–Hölder series of the module associated with the embedding* Φ *is*

$$0 = V_0 \subset V_1 \subset V_2 = \mathbb{C}^{n+2}_{\Phi} \qquad \textit{with } \dim(V_2/V_1) = n+1. \tag{2.8}$$

While that of the module associated to the embedding Θ *is*

$$0 = W_0 \subset W_1 \subset W_2 = \mathbb{C}^{n+2}_{\Theta} \qquad \textit{with } \dim(W_2/W_1) = 1. \tag{2.9}$$

As a consequence of such remark we have

Proposition 2.7 *Let* Ξ *be the automorphims of the rooth space* Δ *of* $\mathfrak{sl}(n+2)$ *given by the relations*

$$\Xi(\alpha_i) = \alpha_{n+2-i} \qquad 1 \le i \le n+1.$$

Denote also by Ξ *the corresponding automorphism of* $\mathfrak{sl}(n+2)$ *generated by the relations*

$$\Xi(H_{\alpha_i}) = H_{\Xi(\alpha_i)} \quad \Xi(E_{\alpha_i}) = E_{\Xi(\alpha_i)} \quad \Xi(F_{\alpha_i}) = F_{\Xi(\alpha_i)} \qquad 1 \le i \le n = 1$$

on the Cartan basis of $\mathfrak{sl}(n+2)$*.*
Then the embedding $\Xi \circ \Theta$, *and* Φ *of* $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ *in* $\mathfrak{sl}(n+2)$ *are equivalent.*

Proof It is easy to show that the map Ξ is the restriction on $\mathfrak{sl}(n+2)$ of the automorphism of the space of the $(n+2)\times(n+2)$ complex matrices given by the equations:

$$\Xi(A_{i,j}) = (-1)^{j-i+1} A_{n+3-j,n+3-i} \qquad 1 \leq i, j \leq n+2.$$

This implies that

$$\Xi\circ\Theta(H_i) = -A_{n+1-i,n+1-i} + A_{n+2-i,n+2-i} \quad 1 \leq i \leq n$$

$$\Xi\circ\Theta(E_{p.q}) = (-1)^{q-p} A_{n+2-p,n+1-q}, \quad \Xi\circ\Theta(F_{p.q}) = (-1)^{q-p} A_{n+1-q,n+2-p} \quad 1 \leq p \leq q \leq n$$

$$\Xi\circ\Theta(P_j) = (-1)^{j+1} A_{n+2-j,n+2} \quad 1 \leq j \leq n+1.$$

Hence the Jordan–Hölder series of the $n+2$ dimensional $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–module associated with this embedding is

$$0 = Z_0 \subset Z_1 \subset Z_2 = \mathbb{C}^{n+2}_{\Theta\circ\Xi} \qquad \text{with } \dim(Z_2/Z_1) = n+1.$$

This latter equation together with equation (2.8) show the equivalence of the embedding $\Xi\circ\Theta$ and Φ.

$\square$

3 Cyclic $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–modules

Let $\mathfrak{g}$ be a Lie algebra, a module M of $\mathfrak{g}$ is said indecomposable if it can not decomposed in the direct sum of two non trivial modules. An important class of indecomposable modules is given by

Definition 3.1 *A module M of a Lie algebra $\mathfrak{g}$ is said cyclic if there exists an element $v \in M$ which generates M, i.e., $M = \mathcal{U}(\mathfrak{g})v$.*

Obviously cyclic modules are indecomposable. Beyond the irreducible modules examples of cyclic modules are the string modules of the Euclidean Lie algebra in two dimensions [18], the modules of the Diamond Lie algebra obtained by embedding it in $\mathfrak{sl}(3,\mathbb{C})$ [1], the uniserial modules [4], and the modules of $\mathfrak{sl}(2,\mathbb{C})\ltimes\mathbb{C}^2$ classified in [16].

Let $\mathfrak{p}$ be the subalgebra of $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–modules, spanned by the elements $\{P_i\}$ $1 \leq i \leq n+1$ (2.3), then since $\mathfrak{p}$ is an ideal in $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$ it holds

Lemma 3.2 *Let M be a $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–module and N be a $\mathfrak{sl}(n+1)$–module contained in M. Then the space $\mathcal{U}(\mathfrak{p})N$ is a $\mathfrak{sl}(n+1)$–submodule.*

Lemma 3.3 *In any finite dimensional $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–module M, the elements P_i, $1 \leq i \leq n+1$ act as nilpotent operators and there exists a positive integer m such that M is annihilated by any monomial of the type $P_{n+1}^{a_{n+1}}\cdots P_1^{a_1}$ with $a_{n+1}+\cdots+a_1 \geq m$.*

Proof Viewed as $\mathfrak{sl}(n+1)$–modules, M can be decomposed in the direct sum of its weight spaces $M = \bigoplus V_\mu$. Now from the commutations relations $\left[H_i, P_j\right] = \delta_{ij}P_j - \delta_{i,j-1}P_j$, $1 \leq i \leq n$, $1 \leq j \leq n+1$ (2.5) it follows that the elements P_i act on the weight space V_μ as

$$P_i V_\mu \longrightarrow V_{\mu+\omega_i-\omega_{i-1}}$$

and therefore on any finite dimensional module they are nilpotent operators. Finally since the operators P_i commute among themselves any monomial of the type $P_{n+1}^{a_{n+1}} \cdots P_1^{a_1}$ is a nilpotent operator, and this implies the second statement of the Lemma.

□

Corollary 3.4 *A $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module M is irreducible if and only if $\mathfrak{p}$ acts on M trivially and M is an irreducible $\mathfrak{sl}(n+1)$–module M.*

This latter result actually is a consequence of the general fact that in a perfect Lie algebra $\mathfrak{g}$, its solvable radical $\mathfrak{r}$ coincides with its nilpotent radical $[\mathfrak{g}, \mathfrak{r}]$ [4].

Any finite dimensional $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module M can be decomposed in a direct sum of irreducible finite dimensional $\mathfrak{sl}(n+1)$–modules:

$$M = \bigoplus_{\lambda\in\Lambda} \pi(\lambda)V(\lambda) \tag{3.1}$$

where $\pi(\lambda)$ is the multiplicity of $V(\lambda)$ in M.

In order to classify the cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–modules we need to study how the irreducible finite dimensional $\mathfrak{sl}(n+1)$–modules are "intertwined" by the action of the ideal $\mathfrak{p}$.

Definition 3.5 *Let $\mu = \sum_{i=1}^{n} \mu_i\omega_i \in P$ be a integral weight of $\mathfrak{sl}(n+1)$, for any pair of integer numbers (k, s) with $1 \leq k \leq s \leq n$ we set*

$$\mu_{k,s} = (-1)\sum_{i=k}^{s}(\mu_i + 1) = -(\mu_s + \mu_{s-1} + \cdots + \mu_k + s - k + 1). \tag{3.2}$$

By the definition of dominant weight and that of $\mu_{k,s}$ follows

Lemma 3.6 *If μ is a dominant weight then for any pair of positive integer numbers (k, s) with $1 \leq k \leq s \leq n$ it holds $\mu_{k,s} > 0$.*

Definition 3.7 *For any integral weight $\mu \in P$, and any integer s, $1 \leq s \leq n$ we set*

$$P_i^{(\mu(s))} = \left(\prod_{k=1}^{i-1} \mu_{k,s-1}\right) P_i \quad 1 \leq i \leq s. \tag{3.3}$$

Observe that if v_μ is a weight vector of weight μ then $P_i^{(\mu(s))}(v_\mu)$ is, if different from zero, a weight vector of weight $\nu = \mu + \omega_i - \omega_{i-1}$, $1 \leq i \leq n$.

Further let for any integral weight $\mu \in P$, and any integer s, we set

$$
\begin{aligned}
F^{(\mu(s))}_{i,j} &= \left(\prod_{k=i+1}^{j} \mu_{k,s}\right) F_{ij} && 1 \le i \le j \le s \\
F^{(\mu(s))}_{i,i} &= F^{(\mu(s))}_{i} = F_i && 1 \le i \le n.
\end{aligned}
\tag{3.4}
$$

Using this formalism let us give

Definition 3.8 *For any integral weight $\mu \in P$, any integer numbers s, k with $1 \le s \le n$, $0 \le k < s$ let $Q^{(\mu(s))}_{s-k}$ be the element of $\mathcal{U}(\mathfrak{sl}(n+1))$ defined by the recurrence relations:*

$$
\begin{aligned}
Q^{(\mu(s))}_{s+1} &= I \\
Q^{(\mu(s))}_{s-k} &= \sum_{l=0}^{k} Q^{(\mu(s))}_{s-l+1} F^{(\mu(s))}_{s-k,s-l} && 0 \le k \le s-1
\end{aligned}
\tag{3.5}
$$

where I is the identity of $\mathcal{U}(\mathfrak{sl}(n+1))$

For example for $s = 3$ we have:

$$
\begin{aligned}
Q^{(\mu(3))}_{4} &= I \\
Q^{(\mu(3))}_{3} &= F^{(\mu(3))}_{3} \\
Q^{(\mu(3))}_{2} &= F^{(\mu(3))}_{2,3} + F^{(\mu(3))}_{3} F^{(\mu(3))}_{2} \\
Q^{(\mu(3))}_{1} &= F^{(\mu(3))}_{1,3} + F^{(\mu(3))}_{3} F^{(\mu(3))}_{1,2} + F^{(\mu(3))}_{2,3} F^{(\mu(3))}_{1} + F^{(\mu(3))}_{3} F^{(\mu(3))}_{2} F^{(\mu(3))}_{1}.
\end{aligned}
$$

Definition 3.9 *For any integral weight $\mu \in P$, any integer numbers s, i, k, with $1 \le s \le n$, $1 \le i \le s$, $0 \le k \le s-i+1$, let $R^{(\mu(s))}_{i,k}$ be the element of $\mathcal{U}(\mathfrak{sl}(n+1))$ defined by the recurrence relations:*

$$
\begin{aligned}
R^{(\mu(s))}_{i,0} &= I && 1 \le i \le s \\
R^{(\mu(s))}_{i,k} &= \sum_{l=0}^{k-1} F^{(\mu(s))}_{i+l,i+k-1} R^{(\mu(s))}_{i,l} && 1 \le i \le s \qquad 1 \le k \le s-i+1
\end{aligned}
\tag{3.6}
$$

where I is the identity of $\mathcal{U}(\mathfrak{sl}(n+1))$.

For example for $s > 3$ and $1 \le i \le s$ we have:

$$
\begin{aligned}
R^{(\mu(s))}_{i,0} &= I \\
R^{(\mu(s))}_{i,1} &= F^{(\mu(s))}_{i} \\
R^{(\mu(s))}_{i,2} &= F^{(\mu(s))}_{i,i+1} + F^{(\mu(s))}_{i+1} F^{(\mu(s))}_{i} \\
R^{(\mu(s))}_{i,3} &= F^{(\mu(s))}_{i,i+2} + F^{(\mu(s))}_{i+1,i+2} F^{(\mu(3))}_{i} + F^{(\mu(s))}_{i+2} F^{(\mu(s))}_{i,i+1} + F^{(\mu(s))}_{i+2} F^{(\mu(s))}_{i+1} F^{(\mu(s))}_{i}.
\end{aligned}
$$

Using the definitions of $P^{(\mu(s))}_{i}$, $Q^{(\mu(s))}_{k}$ $R^{(\mu(s))}_{i,k}$ it is easy to show

Proposition 3.10 *Let $\mu \in P$ be an integral weight. Let s, k be integer numbers, such that $1 \leq s \leq n$, $0 \leq k < s$. Then*

1.
$$Q_k^{(\mu(s))} = \sum_{l=k}^{s} Q_{l+1}^{(\mu(s))} F_{k,l}^{(\mu(s))}. \tag{3.7}$$

2. *$Q_k^{(\mu(s))}$ is a polynomial in $F_{p,q}^{(\mu(s))}$ with $k \leq p \leq q \leq s$, whose powers are at most one.*

3.
$$Q_k^{(\mu(s))} = \sum_{l=1}^{2^{s-k}} M_{k,l}^{(\mu(s))} \qquad 1 \leq k \leq s \tag{3.8}$$

   with $M_{k,l}^{(\mu(s))}$ monomials in $F_{p,q}^{(\mu(s))}$, $k \leq p \leq q \leq s$ such that for any pair of integer numbers (i, j), with $k \leq i \leq j \leq s$ there exists at least one monomial $M_{k,l}^{(\mu(s))}$ such that $F_{i,j}$ divides $M_{k,l}^{(\mu(s))}$.

4. *For any $1 \leq s \leq n$, any $1 \leq k \leq s$, and any $1 \leq l \leq 2^{s-k}$ there exists an integer h with $k \leq h \leq s$ such that $F_{h,s}$ divides $M_{k,l}^{(\mu(s))}$*

5. *If both $F_{p,q}$ and $F_{i,j}$, with $k \leq p \leq q \leq s$ and $k \leq i \leq j \leq s$ divide $M_{k,l}^{(\mu(s))}$ then either $p \leq q < i \leq j$ or $i \leq j < p \leq q$.*

6. *For any monomial $M_{k,l}^{(\mu(s))}$ in equation (3.8) there exist integer numbers $n \geq j_1 \geq j_2 \geq \cdots \geq j_h \geq 1$ such that*

$$M_{k,l}^{(\mu(s))} = \prod_{i=1}^{h+1} \left(\prod_{l=j_i+1}^{j_{i-1}} \mu_{l,s} \right) F_{j_1,s} F_{j_2,j_1-1} \cdots F_{j_h,j_{h-1}-1} F_{k,j_h-1}$$

   where we have set as before $F_{j,0} = 1$ and $j_0 = s$.

Proof

1. It is nothing else but equation (3.5) written in a more convenient way.
2. It is an immediate consequence of the very form of equation (3.7) and of the definition of the elements $Q_k^{(\mu(s))}$, because the elements $F_{p,q}^{(\mu(s))}$ with $p \leq q \leq s$ are not factors of the monomials which appear in the expression of $Q_k^{(\mu(s))}$ with $k > q$.
3. It follows by induction. For $k = s$ the claim is clear. Suppose now that the claim is true for $Q_h^{(\mu(s))}$ with $h > k$ then from (3.7) follows that it is true for $Q_k^{(\mu(s))}$, because if the index i, in $F_{i,j}^{(\mu(s))}$ is $i > k$ then it divides by induction at least one of the monomials of $Q_h^{(\mu(s))}$ with $h > k$, while if $i = k$, $F_{k,j}^{(\mu(s))}$ appears explicitly in (3.7).
4. It follows immediately by induction.
5. Again it follows by induction. For $k = s$ it is clear. Now if the claim is true for $Q_k^{(\mu(s))}$ with $k > j$ then from (3.7) and (2) follows that $F_{j,l}$ for any l with $j \leq l \leq s$ appears as a factor only in monomials of $Q_j^{(\mu(s))}$ in which any other factor $F_{p,q}$ has $j \leq l < p \leq q$.

6. It follows almost directly from the previous points.

□

Proposition 3.11 *Let $\mu \in P$ an integral weight. Let s, i, h integer numbers, such that $1 \le s \le n$, $1 \le i \le s$, $1 \le h \le s - i + 1$. Then*

1.

$$R^{(\mu(s))}_{i,h} = \sum_{l=0}^{h-1} F^{(\mu(s))}_{i+l,i+h-1} R^{(\mu(s))}_{i,l}. \tag{3.9}$$

2. $R^{(\mu(s))}_{i,h}$ is a polynomial in $F^{(\mu(s))}_{p,q}$ with $i \le p \le q \le i + h - 1$, whose power is at most one.

3.

$$R^{(\mu(s))}_{i,h} = \sum_{l=1}^{2^{h-1}} N^{(\mu(s))}_{i,h,l}$$

with $N^{(\mu(s))}_{i,h,l}$ monomials in $F^{(\mu(s))}_{p,q}$, $i \le p \le q \le i + h - 1$ such that for any pair (j, m), $i \le j \le m \le i + h - 1$ there exist at least one monomial $N^{(\mu(s))}_{i,h,l}$ such that $F_{j,m}$ divides $N^{(\mu(s))}_{i,h,l}$.

4. For any $1 \le s \le n$, any $1 \le i \le s$, any $1 \le h \le s - i + 1$, and any $1 \le l \le 2^{h-1}$ there exists an integer m with $k \le m \le i + h - 1$ such that $F_{m,i+h-1}$ divides $N^{(\mu(s))}_{i,h,l}$

5. If both $F_{p,q}$ and $F_{j,m}$, $k \le p \le q \le i + h - 1$, $k \le j \le m \le i + h - 1$ divide $N^{(\mu(s))}_{i,h,l}$ then either $p \le q < j \le m$ or $j \le m < p \le q$.

6. $R^{(\mu(s))}_{i,s-i+1} = Q^{(\mu(s))}_i, \qquad 1 \le i \le s.$

7. If v_μ is a weight vector of weight μ then the vector $R^{(\mu(s-1))}_{i,h} P^{(\mu(s))}_i (v_\mu)$, if not equal to zero, is a weight vector of weight $\mu + \omega_{i+h} - \omega_{i+h-1}$.

Proof Mutata mutandis, for the first five points we can argue as in the previous Proposition. Point 6. follows immediately from the Definitions 3.10 and 3.11; point 7. from the properties of the operators P_i and the commutation rules (2.5).

□

Furthermore using the commutation rules of $\mathfrak{sl}(n + 1) \ltimes \mathbb{C}^{n+1}$, the Definitions 2.5, the equations (3.3) and (3.4) and the above Propositions 3.8 3.9, we have

Proposition 3.12 *Let $\mu \in P$ be an integral weight, $1 \le s \le n$. Then for any $1 \le j \le s$ we have*

$$\begin{array}{llll}
\left[E_j, F^{(\mu(s))}_{j,l}\right] = -\mu_{j+1,s} F^{(\mu(s))}_{j+1,l} & 1 \le j < l & \left[E_j, F^{(\mu(s))}_{l,j}\right] = \mu_{j,s} F^{(\mu(s))}_{l,j-1} & 1 \le l < j \\
\left[E_j, F^{(\mu(s))}_{j}\right] = H_j & & \left[E_j, Q^{(\mu(s))}_{l}\right] = 0 & j < l,\ j > s \\
\left[E_j, R^{(\mu(s))}_{i,h}\right] = 0 & i > j,\ i + h \le j & \left[E_j, P^{(\mu(s))}_{i}\right] = \delta_{j,i-1} \mu_{j,s-1} P^{(\mu(s))}_{j} & 1 \le i \le s.
\end{array} \tag{3.10}$$

Armed with these facts we can prove the

Theorem 3.13 *Let V be a $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module, and let $v_\mu \in V$ be a highest weight vector of $\mathfrak{sl}(n+1)$ of weight μ. Then for $1 \le i \le n+1$ the element*

$$\varphi_i(v_\mu) = P_i^{(\mu(i))}(v_\mu) + \sum_{k=1}^{i-1} Q_k^{(\mu(i-1))} P_k^{(\mu(i))}(v_\mu) \tag{3.11}$$

is either zero or a highest weight vector of $\mathfrak{sl}(n+1)$ of weight $\mu + \omega_i - \omega_{i-1}$.

Proof It is enough to show that for any j with $1 \le j \le n$

$$E_j\, \varphi_i(v_\mu) = 0 \qquad 1 \le i \le n+1. \tag{3.12}$$

Now from the Propositions 3.10, 3.11 and 3.12 and the definition of $\varphi_i(v_\mu)$ it is obvious that

$$E_j\, \varphi_i(v_\mu) = 0 \qquad \text{for } j \ge i.$$

Hence we need only to consider the elements $E_j\varphi_i(v_\mu)$ with $j \le i-1$. Using (3.11) we can write $\varphi_i(v_\mu)$ in this case as

$$\varphi_i(v_\mu) = \sum_{k=j+2}^{i} Q_k^{(\mu(i-1))} P_k^{(\mu(i))}(v_\mu) + \sum_{k=1}^{j+1} Q_k^{(\mu(i-1))} P_k^{(\mu(i))}(v_\mu).$$

Again using propositions 3.10, 3.11 and 3.12 we have

$$E_j\left(\sum_{k=j+2}^{i} Q_k^{(\mu(i-1))} P_k^{(\mu(i))}(v_\mu)\right) = 0.$$

So we need only to consider the action of E_j on $\sum_{k=1}^{j+1} Q_k^{(\mu(i-1))} P_k^{(\mu(i))}(v_\mu)$.

From the formulas (3.10) it follows that if E_j annihilates $\sum_{k=1}^{j+1} Q_k^{(\mu(i-1))} P_k^{(\mu(i))}(v_\mu)$ then it must hold

$$E_j\left(Q_{j+1}^{(\mu(i-1))} P_{j+1}^{(\mu(i))}(v_\mu) + Q_j^{(\mu(i-1))} P_j^{(\mu(i))}(v_\mu)\right) = 0 \tag{3.13}$$

$$E_j\left(Q_k^{(\mu(i-1))} P_k^{(\mu(i))}(v_\mu)\right) = 0 \quad 1 \le k < j. \tag{3.14}$$

Let us start by considering equation (3.13). We have two possible cases: either $i = j+1$ or $i > j+1$. In first case (3.13) becomes

$$E_j\left(P_{j+1}^{(\mu(j+1))}(v_\mu) + F_j^{(\mu(j))} P_j^{(\mu(j+1))}(v_\mu)\right) = 0$$

and we have using equations (3.10)

$$\begin{aligned}
&E_j\left(P_{j+1}^{(\mu(j+1))}(v_\mu) + F_j^{(\mu(j))} P_j^{(\mu(j+1))}(v_\mu)\right)\\
&= E_j\left(P_{j+1}^{(\mu(j+1))}(v_\mu)\right) + \left[E_j, F_j^{(\mu(j))}\right] P_j^{(\mu(j+1))}(v_\mu) = -(\mu_j+1) P_j^{(\mu(j+1))}(v_\mu) + H_j P_j^{(\mu(j+1))}(v_\mu)\\
&= -(\mu_j+1) P_j^{(\mu(j+1))}(v_\mu) + (\mu_j+1) P_j^{(\mu(j+1))}(v_\mu) = 0.
\end{aligned}$$

While if $i > j+1$, equation (3.13) becomes

$$E_j\left(Q_{j+1}^{(\mu(i-1))}P_{j+1}^{(\mu(i))}(v_\mu) + Q_j^{(\mu(i-1))}P_j^{(\mu(i))}(v_\mu)\right) = 0.$$

Now using (3.7) we have

$$\begin{aligned}
Q_{j+1}^{(\mu(i-1))} &= \textstyle\sum_{l=j+1}^{i-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}\\
Q_j^{(\mu(i-1))} &= \textstyle\sum_{l=j}^{i-1} Q_{l+1}^{(\mu(i-1))}F_{j,l}^{(\mu(i-1))}\\
&= \textstyle\sum_{l=j+1}^{i-1} Q_{l+1}^{(\mu(i-1))}F_{j,l}^{(\mu(i-1))} + \sum_{l=j+1}^{i-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}F_j^{(\mu(i-1))}
\end{aligned}$$

therefore using (3.2), (3.10) and Proposition 3.11

$$\begin{aligned}
&E_j\left(Q_{j+1}^{(\mu(i-1))}P_{j+1}^{(\mu(i))}(u_\mu) + Q_j^{(\mu(i-1))}P_j^{(\mu(i))}(v_\mu)\right)\\
&= \textstyle\sum_{l=j+1}^{i-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}\left[E_j, P_{j+1}^{(\mu(i))}\right](v_\mu) + \sum_{l=j+1}^{i-1} Q_{l+1}^{(\mu(i-1))}\left[E_j, F_{j,l}^{(\mu(i-1))}\right]P_j^{(\mu(i))}(v_\mu)\\
&+ \textstyle\sum_{l=j+1}^{i-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}\left[E_j, F_j^{(\mu(i-1))}\right]P_j^{(\mu(i))}(v_\mu)\\
&= \mu_{j,i-1}\textstyle\sum_{l=j+1}^{i-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}P_j^{(\mu(i))}(v_\mu) - \mu_{j+1,i-1}\sum_{l=j+1}^{i-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}P_j^{(\mu(i))}(v_\mu)\\
&+(\mu_j+1)\textstyle\sum_{l=j+1}^{i-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}P_j^{(\mu(i))}(v_\mu) = 0.
\end{aligned}$$

It remains to prove equation (3.14). This requires some more formalism.

Recall that from equation (3.8) $Q_k^{(\mu(s))}$ for $1 \le k \le s$ is the sum of monomials $M_{k,l}^{(\mu(s))}$ with $1 \le l \le 2^{s-k}$. For any j, $k \le j \le s$ let $J_{k,s}^+(j)$ be the set:

$$J_{k,s}^+(j) = \left\{1 \le l \le 2^{s-k} \;\middle|\; \exists\, h\ j \le h \le s \text{ such that } F_{j,h} \text{ divides } M_{k,l}^{(\mu(s))}\right\}$$

i.e., the set of all l, $1 \le l \le 2^{s-k}$ such that the monomial $M_{k,l}^{(\mu(s))}$ contains a factor of type $F_{j,h}$ with $j \le h \le s$. Let $Q_{k,j}^{(\mu(s))^+}$ be defined by

$$Q_{k,j}^{(\mu(s))^+} = \sum_{l\in J_{k,s}^+(j)} M_{k,l}^{(\mu(s))}, \tag{3.15}$$

then from proposition 3.10 anf 3.11 follows almost immediately that for any $1 \le j \le n$

$$Q_{k,j}^{(\mu(s))^+} = Q_j^{(\mu(s))}R_{k,j-k}^{(\mu(s))}. \tag{3.16}$$

Now from (3.7) and (3.9) we have

$$\begin{aligned}
Q_j^{(\mu(s))} &= \textstyle\sum_{l=j}^{s} Q_{l+1}^{(\mu(s))}F_{j,l}^{(\mu(s))}\\
R_{k,j-k}^{(\mu(s))} &= \textstyle\sum_{l=0}^{j-k-1} F_{k+l,j-1}^{(\mu(s))}R_{k,l}^{(\mu(s))}
\end{aligned}$$

using these latter equations in (3.16) we have

$$Q_{k,j}^{(\mu(s))^+} = \sum_{l=j}^{s} \sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(s))} F_{j,l}^{(\mu(s))} F_{k+h,j-1}^{(\mu(s))} R_{k,h}^{(\mu(s))}. \tag{3.17}$$

We still have to take into account those monomials in the expression of $Q_k^{(\mu(s))}$ which are divisible for $F_{h,j}$ with $k \le h < j$ (those with $h = j$ has been already considered).

As above, for any j, $k \le j \le s$ let $J_{s,k}^-(j)$ be the set:

$$J_{k,s}^-(j) = \left\{ 1 \le l \le 2^{s-k} \;\middle|\; \exists\, h,\; k \le h < j \text{ such that } F_{h,j} \text{ divides } M_{k,l}^{(\mu(s))} \right\}$$

i.e., the set of all l, $1 \le l \le 2^{s-k}$ such that the monomial $M_{k,l}^{(\mu(s))}$ contains a factor of type $F_{h,j}$ with $k \le h < j$. Let $Q_{k,j}^{(\mu(s))^-}$ be defined by

$$Q_{k,j}^{(\mu(s))^-} = \sum_{l \in J_{k,s}^-(j)} M_{k,l}^{(\mu(s))}. \tag{3.18}$$

Arguing as in the previous case we have that

$$Q_{k,j}^{(\mu(s))^-} = Q_{j+1}^{(\mu(s))} \left(R_{k,j-k+1}^{(\mu(s))} - \text{ the monomials in } R_{k,j-k+1}^{(\mu(s))} \text{ which are divisible by } F_j \right). \tag{3.19}$$

Now from Proposition 3.11 and equation (3.9) it follows that

$$\left(R_{k,j-k+1}^{(\mu(s))} - \text{ the monomials in } R_{k,j-k+1}^{(\mu(s))} \text{ which are divisible by } F_j \right) = \sum_{h=0}^{j-k-1} F_{k+h,j}^{(\mu(s))} R_{k,h}^{(\mu(s))}.$$

Therefore using this latter equation together with (3.7) in (3.19) we have

$$Q_{k,j}^{(\mu(s))^-} = \sum_{l=j+1}^{s} \sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(s))} F_{j+1,l}^{(\mu(s))} F_{k+h,j}^{(\mu(s))} R_{k,h}^{(\mu(s))}. \tag{3.20}$$

Now, for what said above, since obviously $\left[E_j, M_{k.l}^{(\mu(i-1))} \right] = 0$ if $M_{k.l}^{(\mu(i-1))}$ is not divisible by $F_{h,j}$, $k \le h \le j$ or $F_{j,h}$ $j \le h \le i-1$ we have only to show that

$$E_j(Q_{k,j}^{(\mu(i-1))^+} P_k^{(\mu(i))}(v_\mu) + Q_{k,j}^{(\mu(i-1))^-} P_k^{(\mu(i))}(v_\mu)) = 0.$$

Taking into account equations (3.17), (3.20) and Proposition 3.11 we have

$$
\begin{aligned}
&E_j\Big(\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}F_{j,l}^{(\mu(i-1))}F_{k+h,j-1}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\\
&+\textstyle\sum_{h=0}^{j-k-1} Q_{j+1}^{(\mu(i-1))}F_{j}^{(\mu(i-1))}F_{k+h,j-1}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\\
&+\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}F_{k+h,j}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\Big)\\
&=E_j\Big(\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}F_{j,l}^{(\mu(i-1))}F_{k+h,j-1}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\\
&+\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}F_{j}^{(\mu(i-1))}F_{k+h,j-1}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\\
&+\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}F_{k+h,j}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\Big)\\
&=\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}\Big[E_j,F_{j,l}^{(\mu(i-1))}\Big]F_{k+h,j-1}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\\
&+\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}\Big[E_j,F_{j}^{(\mu(i-1))}\Big]F_{k+h,j-1}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\\
&+\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}\Big[E_j,F_{k+h,j}\Big]R_{h,k}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\\
&=-\mu_{j+1,i-1}\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}F_{k+h,j-1}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\\
&+(\mu_j+1)\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}F_{k+h,j-1}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)\\
&+\mu_{j,i-1}\textstyle\sum_{l=j+1}^{i-1}\sum_{h=0}^{j-k-1} Q_{l+1}^{(\mu(i-1))}F_{j+1,l}^{(\mu(i-1))}F_{k+h,j-1}^{(\mu(i-1))}R_{k,h}^{(\mu(i-1))}P_k^{(\mu(i))}(v_\mu)=0.
\end{aligned}
$$

□

Remark 3.14 *The first elements $\varphi_i(v_\mu)$ are*

$$
\begin{aligned}
\varphi_1(v_\mu) &= P_1v_\mu\\
\varphi_2(v_\mu) &= -(\mu_1+1)P_2v_\mu+F_1P_1v_\mu\\
\varphi_3(v_\mu) &= (\mu_2+1)(\mu_2+\mu_1+2)P_3v_\mu-(\mu_2+\mu_1+2)F_2P_2v_\mu\\
&-(\mu_2+1)F_{1,2}P_1v_\mu+F_2F_1P_1v_\mu\\
\varphi_4(v_\mu) &= -(\mu_3+1)(\mu_3+\mu_2+2)(\mu_3+\mu_2+\mu_1+3)P_4v_\mu\\
&+(\mu_3+\mu_2+2)(\mu_3+\mu_2+\mu_1+3)F_3P_3v_\mu+(\mu_3+1)(\mu_3+\mu_2+\mu_1+3)F_{2,3}P_2v_\mu\\
&-(\mu_3+\mu_2+\mu_1+3)F_3F_2P_2v_\mu+(\mu_3+1)(\mu_3+\mu_2+2)F_{1,3}P_1v_\mu\\
&-(\mu_3+\mu_2+2)F_3F_{1,2}P_1v_\mu-(\mu_3+1)F_{2,3}F_1P_1v_\mu+F_3F_2F_1P_1v_\mu.
\end{aligned}
$$

Corollary 3.15 *Let V be a $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–module, $V(\mu)\subset V$ be an irreducible $\mathfrak{sl}(n+1)$–module of highest weight μ and highest vector v_μ, then for any $1\le i\le n+1$ there exists a positive integer k_i such that*

1. $\varphi_i^{k_i}(v_\mu) = 0$
2. *For any i,* $1 \le i \le n+1$ *and any k,* $1 \le k < k_i$*, the space* $\mathcal{U}(\mathfrak{sl}(n+1))\varphi_i^k(v_\mu)$ *is a finite irreducible* $\mathfrak{sl}(n+1)$*–module with highest weight* $\mu + k(\omega_i - \omega_{i-1})$ *and highest weight vector* $\varphi_i^k(v_\mu)$*.*

Proof That any $\varphi_i^k(v_\mu) \neq 0$, $1 \le i \le n+1$, $k \in \mathbb{N}$, is a highest weight vector for $\mathfrak{sl}(n+1)$ of weight $\mu + k(\omega_i - \omega_{i-1})$ has been already proved in the previous Theorem 3.13.

The existence of a such k_i for any $1 \le i \le n+1$ follows form the fact that if different from zero $\varphi_i^k(v_\mu)$ and $\varphi_i^h(v_\mu)$ have different weight if $k \neq h$ and that the space V is finite dimensional. The implication $\varphi_i^k(v_\mu) = 0 \Longrightarrow \varphi_i^h(v_\mu) = 0$ if $k \le h$, $1 \le i \le n+1$ proves the second statement of the Corollary.

□

The formulas (3.11) may be inverted, let us give indeed:

Definition 3.16 *For any integral dominant weight* $\mu \in \Lambda$ *any positive integer numbers j, k, i with* $1 \le j \le k \le n$ *and* $1 \le i \le k$ *define*

$$\widehat{F}_{j.k}^{(\mu,i)} = \frac{1}{|\mu_{i,k}|} F_{j,k} \tag{3.21}$$

where $|x|$ *is the absolute value of x.*

Definition 3.17 *For any integral dominant weight* $\mu \in \Lambda$*, any positive integer numbers s, i, k with* $1 \le i \le s \le n$ *and* $i \le k \le s+1$ *let* $\widehat{Q}_k^{(\mu(s),i)}$ *be the element of* $\mathcal{U}(\mathfrak{sl}(n+1))$ *defined by the recurrence relation:*

$$\begin{aligned} \widehat{Q}_{s+1}^{(\mu(s),i)} &= I \\ \widehat{Q}_{k}^{(\mu(s),i)} &= \sum_{l=k+1}^{s+1} \widehat{Q}_{l}^{(\mu(s),i)} \widehat{F}_{k,l-1}^{(\mu,i)} \qquad i \le k < s+1. \end{aligned} \tag{3.22}$$

Observe that from Lemma 3.6 it follows that the elements $\widehat{F}_{j.k}^{(\mu,i)}$ (3.21) and the following elements $\widehat{\varphi}_i(v_\mu)$ (3.23) as well are well defined. Then it holds

Proposition 3.18 *Let V be a* $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$*–module, and let* $v_\mu \in V$ *be a highest weight vector of* $\mathfrak{sl}(n+1)$ *of integral dominant weight* μ*. Let further* $\widehat{\varphi}_i(v_\mu)$*,* $1 \le i \le n+1$ *be defined by*

$$\widehat{\varphi}_i(v_\mu) = \frac{1}{\prod_{k=1}^{i-1} |\mu_{k,i-1}|} \varphi_i(v_\mu) \qquad 1 \le i \le n+1. \tag{3.23}$$

Then the elements $P_i v_\mu$*,* $1 \le i \le n+1$ *can be written as*

$$P_i v_\mu = (-1)^{i+1} \widehat{\varphi}_i(v_\mu) + \sum_{k=1}^{i-1} (-1)^{k+1} \widehat{Q}_k^{(\mu(i-1),k)} \widehat{\varphi}_k(v_\mu) \qquad 1 \le i \le n+1. \tag{3.24}$$

Proof The transformation between the vectors $P_i v_\mu$ and the vectors $\varphi_i(v_\mu)$ $i = 1, \dots n+1$ given by the equations (3.11) can be viewed as an upper triangular matrix with entries on the main diagonal different from zero. Therefore it can be inverted.

It remains to show that the explicit form of such inverse is given by equations (3.24). We shall argue by induction over the rank of the simple Lie algebra $\mathfrak{sl}(n+1)$ (i.e., over the integer number n). For $n = 1$ the claim is obvious. We shall show that the equations (3.24) hold for $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ if they hold for $\mathfrak{sl}(n) \ltimes \mathbb{C}^n$.

A moment thought reveals that for $1 \le i \le n$ the explicit form of the element $\varphi_i(v_\mu)$ in a $\mathfrak{sl}(n) \ltimes \mathbb{C}^n$–module is the same of that of the corresponding element in a $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–modules. Hence we need only to prove that equations (3.24) hold when $i = n+1$. From the very same reason we also need only to show that

$$P_{n+1} v_\mu = \sum_{l=2}^{n+1} S_l(F_{p,q}) \varphi_l(v_\mu) + \widehat{Q}_1^{(\mu(n),1)} \widehat{\varphi}_1(v_\mu) \qquad S_l(F_{p,q}) \in \mathcal{U}(\mathfrak{n}^-) \quad 2 \le l \le n+1. \quad (3.25)$$

Using equation (3.11) with $i = n+1$ together (by induction Hypothesis) with equations (3.24) with $i \le n$ and the point 6. of Proposition 3.10 we obtain

$$\begin{aligned} P_{n+1} v_\mu &= \textstyle\sum_{l=2}^{n+1} S_l(F_{p,q}) \varphi_l(v_\mu) \\ &+ \sum_{n \ge k_1 \ge k_2 \ge \cdots \ge k_h \ge 1} \frac{1}{\prod_{l=1}^{n} |\mu_{l,n}|} \sum_{i=1}^{h+1} \left(\frac{(-1)^{i+1} \prod_{l=1}^{n} |\mu_{l,n}|}{\prod_{l=1}^{i} |\mu_{k_l,n}| \prod_{j=i}^{h} |\mu_{1,k_{j-1}}|} \right) F_{n,k_1} F_{k_2,k_1-1} \cdots F_{k_h,k_{h-1}-1} F_{1,k_h-1} \varphi_1(v_\mu) \end{aligned}$$

where the $S_l(F_{p,q})$, $2 \le l \le n+1$ are polynomials in the variable $F_{p,q}$ with $2 \le p \le q \le n$. It is easy to compute that

$$\frac{1}{\prod_{l=1}^{n} |\mu_{l,n}|} \sum_{i=1}^{h+1} \left(\frac{(-1)^{i+1} \prod_{l=1}^{n} |\mu_{l,n}|}{\prod_{l=1}^{i} |\mu_{k_l,n}| \prod_{j=i}^{h} |\mu_{1,k_{j-1}}|} \right) = \frac{1}{\prod_{l=0}^{h} |\mu_{1,k_l-1}|}, \quad k_0 = n+1.$$

Now Definition of $\widehat{Q}_1^{\mu(n),1}$ 3.17 and Proposition 3.10 yield

$$\widehat{Q}_1^{\mu(n),1} = \sum_{n \ge k_1 \ge k_2 \ge \cdots \ge k_h \ge 1} \frac{1}{\prod_{l=0}^{h} |\mu_{1,k_l-1}|} F_{n,k_1} F_{k_2,k_1-1} \cdots F_{k_h,k_{h-1}-1} F_{1,k_h-1}$$

which implies equation (3.25) as wanted.

$\square$

Remark 3.19 *The first examples of equations (3.24) are*

$$\begin{aligned}
P_1(v_\mu) &= \varphi_1(v_\mu)\\
P_2 v_\mu &= -\tfrac{1}{\mu_1+1}\varphi_2(v_\mu) + \tfrac{1}{\mu_1+1}F_1\varphi_1(v_\mu)\\
P_3(v_\mu) &= \tfrac{1}{(\mu_2+1)(\mu_2+\mu_1+2)}\varphi_3(v_\mu) - \tfrac{1}{(\mu_1+1)(\mu_2+1)}F_2\varphi_2(v_\mu)\\
&+\tfrac{1}{\mu_2+\mu_1+2}F_{1,2}\varphi_1(v_\mu) + \tfrac{1}{(\mu_2+\mu_1+2)(\mu_1+1)}F_2F_1\varphi_1(v_\mu)\\
P_4(v_\mu) &= -\tfrac{1}{(\mu_3+1)(\mu_3+\mu_2+2)(\mu_3+\mu_2+\mu_1+3)}\varphi_4(v_\mu) + \tfrac{1}{(\mu_3+1)(\mu_2+1)(\mu_2+\mu_1+2)}F_3\varphi_3(v_\mu)\\
&-\tfrac{1}{(\mu_3+\mu_2+2)(\mu_1+1)}F_{2,3}\varphi_2(v_\mu) - \tfrac{1}{(\mu_3+\mu_2+2)(\mu_2+1)(\mu_1+1)}F_3F_2\varphi_2(v_\mu)\\
&+\tfrac{1}{\mu_3+\mu_2+\mu_1+3}F_{1,3}\varphi_1(v_\mu) + \tfrac{1}{(\mu_3+\mu_2+\mu_1+3)(\mu_2+\mu_1+2)}F_3F_{1,2}\varphi_1(v_\mu)\\
&+\tfrac{1}{(\mu_3+\mu_2+\mu_1+3)(\mu_1+1)}F_{2,3}F_1\varphi_1(v_\mu) + \tfrac{1}{(\mu_3+\mu_2+\mu_1+3)(\mu_2+\mu_1+2)(\mu_1+1)}F_3F_2F_1P_1v_\mu.
\end{aligned}$$

Theorem 3.20 *Let V be a $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–module, $v_\mu \in V$ be an highest vector of $\mathfrak{sl}(n+1)$ of weight μ. Then it holds*

$$\varphi_i(\varphi_j(v_\mu)) = \varphi_j(\varphi_i(v_\mu)) \qquad 1 \le i, j \le n+1 \tag{3.26}$$

Proof Using equations (3.2) (3.11) and the commutation rules (2.5) we have for $1 \le i \le n$

$$\begin{aligned}
&\varphi_{i+1}(\varphi_i(v_\mu)) - \varphi_i(\varphi_{i+1}(v_\mu)) = -(\mu_i+2)\textstyle\prod_{k=1}^{i-1}\mu_{k,i}\prod_{h=1}^{i-1}\mu_{h,i-1}P_{i+1}P_iv_\mu\\
&+\textstyle\prod_{k=1}^{i-1}\mu_{k,i}\prod_{h=1}^{i-1}\mu_{h,i-1}F_iP_i^2v_\mu + \displaystyle\sum_{\substack{s\le r\le i+1\\ r+s\le 2i}} S^{(i+1,i)}_{r,s}P_rP_sv_\mu\\
&+(\mu_i+1)\textstyle\prod_{k=1}^{i-1}\mu_{k,i}\prod_{h=1}^{i-1}\mu_{h,i-1}P_{i+1}P_iv_\mu - \prod_{k=1}^{i-1}\mu_{k,i}\prod_{h=1}^{i-1}\mu_{h,i-1}\left[P_i,F_i\right]P_iv_\mu\\
&-\textstyle\prod_{k=1}^{i-1}\mu_{k,i}\prod_{h=1}^{i-1}\mu_{h,i-1}F_iP_i^2v_\mu + \displaystyle\sum_{\substack{s\le r\le i+1\\ r+s\le 2i}} T^{(i+1,i)}_{r,s}P_rP_sv_\mu\\
&=\sum_{\substack{s\le r\le i+1\\ r+s\le 2i}} W^{(i+1,i)}_{r,s}P_rP_sv_\mu \qquad \text{with}\quad S^{(i+1,i)}_{r,s}, T^{(i+1,i)}_{r,s}, W^{(i+1,i)}_{r,s} \in \mathcal{U}(\mathfrak{n}^-).
\end{aligned}\tag{3.27}$$

While a similar but easier computation shows for $2 \le i \le n$ and $1 \le j < i-1$ that

$$\varphi_{i+1}(\varphi_j(v_\mu)) - \varphi_j(\varphi_{i+1}(v_\mu)) = \sum_{\substack{s\le r\le i+1\\ r+s\le i+j}} W^{(i+1,j)}_{r,s}P_rP_sv_\mu \qquad W^{(i+1,i)}_{r,s} \in \mathcal{U}(\mathfrak{n}^-). \tag{3.28}$$

For $i = 1$ it follows from equation (3.27) that $\phi_2(\phi_1(v_\mu)) = \phi_1(\phi_2(v_\mu))$ (see also Piard [16]). While for $1 < i \le n$, $1 \le j < i$, using equations (3.24) in the equations (3.27) and (3.28) we obtain

$$\varphi_{i+1}(\varphi_j(v_\mu)) - \varphi_j(\varphi_{i+1}(v_\mu)) = \sum_{\substack{s\le r\le i+1\\ r+s\le i+j}} Z^{(i+1,j)}_{r,s}\varphi_r(\varphi_s(v_\mu)) \qquad Z^{(i+1,i)}_{r,s} \in \mathcal{U}(\mathfrak{n}^-). \tag{3.29}$$

Now from Theorem 3.13 it follows that the vector $\varphi_{i+1}(\varphi_j(v_\mu)) - \varphi_j(\varphi_{i+1}(v_\mu))$ if different from zero is a highest weight vector of $\mathfrak{sl}(n+1)$ with weight $\mu_{i+1,j} = \mu+\omega_{i+1}-\omega_i+\omega_j-\omega_{j-1}$, while from equation (3.29) follows that the same vector belongs to the $\mathfrak{sl}(n+1)$–submodule

$$\varphi_{i+1}(\varphi_j(v_\mu)) - \varphi_j(\varphi_{i+1}(v_\mu)) \in \bigoplus_{\substack{s\leq r\leq i+1 \\ r+s\leq i+j}} V(\mu_{r,s}) \qquad \mu_{r,s} = \mu + \omega_r - \omega_{r-1} + \omega_s - \omega_{s-1}.$$

But this is impossible because $i > 1$, $s \leq r \leq i+1$, and $r+s \leq i+j$, imply $\mu_{i+1,j} \neq \mu_{r,s}$. Therefore we must have

$$\varphi_i(\varphi_j(v_\mu)) = \varphi_j(\varphi_i(v_\mu)) \qquad 1 \leq i, j \leq n+1.$$

□

Proposition 3.21 *Let V be a $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–module and $V(\mu)$ a irreducible finite $\mathfrak{sl}(n+1)$–module contained in V of highest weight vector v_μ and highest weight μ. Then for any there exists a k such that $(\varphi_{n+1}\circ\varphi_n\cdots\circ\varphi_1)^k(v_\mu) = 0$ and the $\mathfrak{sl}(n+1)$–modules generates by the highest weight vectors $(\varphi_{n+1}\circ\varphi_n\cdots\circ\varphi_1)^j(v_\mu)$, $1 \leq j < k$ are irreducible finite modules of highest weight μ such that for $1 \leq l < j < k$ it holds*

$$\mathcal{U}(\mathfrak{sl}(n+1))\,(\varphi_{n+1}\circ\varphi_n\cdots\circ\varphi_1)^l(v_\mu) \cap \mathcal{U}(\mathfrak{sl}(n+1))\,(\varphi_{n+1}\circ\varphi_n\cdots\circ\varphi_1)^j(v_\mu) = \{0\}.$$

Proof The existence of a k such that $(\varphi_{n+1}\circ\varphi_n\cdots\circ\varphi_1)^k(v_\mu) = \{0\}$ and $(\varphi_{n+1}\circ\varphi_n\cdots\circ\varphi_1)^j(v_\mu) \neq \{0\}$ for $1 \leq j < k$ follows from Theorem 3.20 and the nilpotency of the operators involved. That $(\varphi_{n+1}\circ\varphi_n\cdots\circ\varphi_1)^j(v_\mu)$ is, if different from zero, a highest weight vector of weight μ follows from Theorem 3.13. While the last statement of the Proposition holds for the same argument used in the proof of Lemma 7 of [16].

□

From Theorem 3.20, the form of the perfect Lie algebra $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$ and the fact that from Proposition 3.18 the action of the radical $\mathfrak{p}$ on any highest weight vector v_μ is determined by the element $\varphi_i(v_\mu)$, $1 \leq i \leq n+1$ we have

Proposition 3.22 *Let V a cyclic $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–module with generator v_{μ_0} then V has the form*

$$V = \bigoplus_{(k_1,\ldots,k_{n+1})\in\mathbb{N}^{n+1}} \mathcal{U}(\mathfrak{sl}(n+1))\varphi_{n+1}^{k_{n+1}}(\varphi_n^{k_n}\cdots(\varphi_1^{k_1}(v_{\mu_0})\cdots))).$$

Since by the Weyl Theorem any cyclic $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–module is the direct sum of finite dimensional irreducible modules of $\mathfrak{sl}(n+1)$–modules we can always suppose that the generator v belongs to an irreducible $\mathfrak{sl}(n+1)$–submodules. Furthermore being such module an irreducible $\mathfrak{sl}(n+1)$–module, we can suppose that v is an highest weight vector of weight say μ_0: $v = v_{\mu_0}$ as well. Such highest weight vector is up multiplicative factor unique.

Proposition 3.23 *Let V be a cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module with generator the $\mathfrak{sl}(n+1)$–highest weight vector v_{μ_0} of weight μ_0 then any other generators of V belongs to the irreducible highest weight $\mathfrak{sl}(n+1)$–module $V(\mu_0) = \mathcal{U}(\mathfrak{sl}(n+1))v_{\mu_0}$.*

Proof It is a direct consequence of Proposition 3.22

□

On behalf of Corollary 3.15 and Theorem 3.20 we can give

Definition 3.24 *Let V be a cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module, generated by the highest $\mathfrak{sl}(n+1)$ weigth vector $v_\mu \in V$ of weight μ.*
Let J_1^μ be the littlest positive integer number such that

$$\varphi_1^{J_1^\mu}(v_\mu) = 0.$$

Further, for any $0 \le i_1 < J_1^\mu$, let $J_2^\mu(i_1)$ be the littlest positive integer such that

$$\varphi_2^{J_2^\mu(i_1)}(\varphi_1^{i_1}(v_\mu)) = 0, \quad 0 \le i_1 < J_1^\mu.$$

Recursively suppose that we have already defined $J_h^\mu(i_1, \dots i_{h-1})$ for $1 \le h \le l$ and $0 \le i_r < J_r^\mu(i_1, \dots i_{r-1})$, $1 \le r \le h$ then $J_{l+1}^\mu(i_1, i_2, \dots, i_l)$ is the littlest positive integer such that

$$\varphi_{l+1}^{J_{l+1}^\mu(i_1,i_2,\dots,i_l)}(\varphi_l^{i_l}(\cdots(\varphi_1^{i_1}(v_\mu)))) = 0, \quad 0 \le i_1 < J_1^\mu, \quad 0 \le i_h < J_h^\mu(i_1, \dots, i_{h-1}) \quad 2 \le h \le l.$$

Hence $J_{n+1}^\mu(i_1, i_2, \dots, i_n)$ is the littlest positive integer such that

$$\varphi_{n+1}^{J_{n+1}^\mu(i_1,i_2,\dots,i_n)}(\varphi_n^{i_n}(\cdots(\varphi_1^{i_1}(v_\mu)))) = 0, \quad 0 \le i_1 < J_1^\mu, \quad 0 \le i_k < J_k^\mu(i_1, \dots i_{k-1}) \quad 2 \le k \le n.$$

Let us finally denote by $\mathcal{J}^\mu$ the set given by all the numbers J_1^μ, $J_k^\mu(i_1, \dots, i_{k-1})$ i.e.,:

$$\mathcal{J}^\mu = \left\{ J_1^\mu, J_k^\mu(i_1, \dots, i_{k-1}) \,\middle|\, 0 \le i_1 < J_1^\mu,\ 0 \le i_k < J_k^\mu(i_1, \dots i_{k-1}) \quad 2 \le k \le n+1 \right\}.$$

Theorem 3.25 *Let V be a cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module, generated by the highest $\mathfrak{sl}(n+1)$ weigth vector $v_\mu \in V$ of weight μ. Then for the elements of the set $\mathcal{J}^\mu$ of definition 3.24 it holds*

1. $J_1^\mu \ge 1$;
2. *the set $\mathcal{J}^\mu$, is bounded;*
3. *for $1 \le k \le n$ it holds*

$$J_k^\mu(i_1, i_2, \dots, i_{k-1}) \le J_k^\mu(j_1, j_2, \dots, j_{k-1}) \ \text{ if } \quad j_l \le i_l \quad 1 \le l \le k-1 \quad 2 < k \le n+1 \tag{3.30}$$

4. $1 \le J_k^\mu(i_1, \dots i_{k-1}) \le \mu_{k-1} + 1 \quad \forall i_h < J_h(i_1, \dots i_{h-1}) \quad 1 \le h \le k \quad 2 < k \le n+1.$

Proof

1. It is obvious form the very definition of J_1^μ that it must be equal or bigger then 1,
2. It is a consequence of Corollary 3.15.
3. By definition

   $$\varphi_l^{J_l^\mu(i_1,\dots i_{l-1})-1}(\varphi_{l-1}^{i_{l-1}}(\cdots(\varphi_k^{i_k}(\cdots(\varphi_1^{i_1}(v_\mu)))) \neq 0 \quad i_k < J_k^\mu(i_1,\dots,i_{k-1}) \quad 1 \leq k \leq l-1.$$

   Now from Theorem 3.20 it follows for any $2 \leq l \leq n$ and any $1 \leq k \leq l-1$ with $i_k > 0$ that

   $$\varphi_l^{J_l^\mu(i_1,\dots i_{l-1})-1}(\varphi_{l-1}^{i_{l-1}}(\cdots(\varphi_k^{i_k}(\cdots(\varphi_1^{i_1}(v_\mu)))) = \varphi_k(\varphi_l^{J_l^\mu(i_1,\dots i_{l-1})-1}(\varphi_{l-1}^{i_{l-1}}(\cdots(\varphi_k^{i_k-1}(\cdots(\varphi_1^{i_1}(v_\mu)))).$$

   This implies

   $$\varphi_l^{J_l^\mu(i_1,\dots i_{l-1})-1}(\varphi_{l-1}^{i_{l-1}}(\cdots(\varphi_k^{i_k-1}(\cdots(\varphi_1^{i_1}(v_\mu)))) \neq 0$$

   and in turn

   $$J_l^\mu(i_1,\cdots,i_k-1,\cdots,i_{l-1}) \geq J_l^\mu(i_1,\cdots,i_k,\cdots,i_{l-1}) \quad 2 \leq l \leq n \quad 1 \leq k \leq l-1, \;\; i_k > 0.$$

4. We need only to prove that $J_k^\mu(i_1,\dots,i_{k-1}) \leq \mu_{k-1}+1$ for $2 \leq k \leq n+1$. Further equation (3.30) implies that it is enough to show that $J_k^\mu(0,\dots,0) \leq \mu_{k-1}+1$ i.e., that

   $$\varphi_k^{\mu_{k-1}+1}(v_\mu) = 0 \quad 2 \leq k \leq n+1.$$

   But from Theorem 3.13 we have that if $\varphi_k^{\mu_{k-1}+1}(v_\mu) \neq 0$ then it is a highest weight vector of weight

   $$\nu = \sum_{l=1}^{k-2} \mu_l \omega_l - \omega_{k-1} + (2\mu_k+1)\omega_k + \sum_{l=k+1}^{n+1} \mu_l \omega_l$$

   which is impossible since ν is not a dominant integral weight of $\mathfrak{sl}(n+1)$.

□

Definition 3.26 *According to Definition 3.24 and using Theorem 3.25 and Proposition 3.23 we can associate to any cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module V the set*

$$\mathcal{J}(V) = \left\{\mu_0, J_1^{\mu_0}, J_k^{\mu_0}(i_1,\dots,i_{k-1}) \;\middle|\; i_1 < J_1^{\mu_0},\; i_k < J_k^{\mu_0}(i_1,\dots i_{k-1}) \quad 2 \leq k \leq n+1\right\} \tag{3.31}$$

where μ_0 is the $\mathfrak{sl}(n+1)$ weight of the highest weight vector v_{μ_0} which generates V, and the numbers $J_1^{\mu_0}$, $J_k^{\mu_0}(i_1,\dots,i_{k-1})$, $2 \leq k \leq n+1$ are those defined in Definition 3.24.

Theorem 3.27 *If two cyclic finite dimensional modules V and W of $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ generated respectively by the highest weight vector v_{μ_0} and w_{ν_0} of weight μ_0 and ν_0 are equivalent then the two set $\mathcal{J}(V)$ and $\mathcal{J}(W)$ given by equation (3.31) coincide.*

Proof From the Propositions 3.22, and 3.23 follows that the modules V and W decomposes as

$$
\begin{aligned}
V = &\quad V(\mu_0) \bigoplus_{k_l\in\mathbb{N},\sum_{l=1}^{n+1} k_l\geq 1} \mathcal{U}(\mathfrak{sl}(n+1))\varphi_{n+1}^{k_{n+1}}(\cdots\varphi_l^{k_l}\cdots(\varphi_1^{k_1}(v_{\mu_0}))))\\
W = &\quad W(\nu_0) \bigoplus_{h_j\in\mathbb{N},\sum_{j=1}^{n+1} h_j\geq 1} \mathcal{U}(\mathfrak{sl}(n+1))\varphi_{n+1}^{h_{n+1}}(\cdots\varphi_l^{h_l}\cdots(\varphi_1^{h_1}(w_{\nu_0}))))
\end{aligned}
$$

here $V(\mu_0)$ (resp. $W(\nu_0)$) is the irreducible $\mathfrak{sl}(n+1)$–module with highest weight μ_0 (res. ν_0) and highest weight vector v_{μ_0} (resp. w_{ν_0}).

If the module V and W are equivalent then there exists an invertible linear operator $T \in \mathrm{Hom}_{\mathbb{C}}(V,W)$ such that $XT = TX$ for any element X in $\mathcal{U}(\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1})$. Hence, since $V(\mu_0) = \mathcal{U}(\mathfrak{sl}(n+1))v_{\mu_0}$ and $W(\nu_0) = \mathcal{U}(\mathfrak{sl}(n+1))w_{\nu_0}$ the Schur Lemma [13] [14] implies that $\mu_0 = \nu_0$ and therefore that $W(\nu_0) \simeq V(\mu_0)$. For any highest weight vector v_μ in V, Tv_μ is an highest weight vector in W with the same weight μ. Hence, since T commutes with the action of $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$ we have $T\varphi_i(u_\mu) = \varphi_i(Tu_\mu)$ for every highest weight vector u_μ and any i, $1 \leq i \leq n+1$. This implies

$$
\varphi_{n+1}^{k_{n+1}}(\cdots\varphi_l^{k_l}\cdots(\varphi_1^{k_1}(v_{\mu_0})))) \neq 0 \Longleftrightarrow \varphi_{n+1}^{k_{n+1}}(\cdots\varphi_l^{k_l}\cdots(\varphi_1^{k_1}(w_{\mu_0}))) \neq 0.
$$

This latter equation obviously shows that set $\mathcal{J}(V)$ and $\mathcal{J}(W)$ coincide.

□

In order to complete the classification of all cyclic $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–modules, it remains to show that for any set $\mathcal{M}$ which satisfies the requirements of Definition 3.26 together with those of Theorem 3.25 there exists a cyclic $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–module V such that $\mathcal{J}(V) = \mathcal{M}$.

This will be done in the next section where we shall show how such modules are quotient modules of the restriction to $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$ of finite dimensional irreducible $\mathfrak{sl}(n+2)$–modules.

4 The $\mathfrak{sl}(n+2)$–modules as $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–modules

Viewed in the light of the previous section the embeddings Φ and Θ of $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$ in $\mathfrak{sl}(n+2)$ given in Theorem 2.5 by the formulas (2.6) and (2.7) are $\mathfrak{sl}(n+1)\ltimes\mathbb{C}^{n+1}$–modules $\mathbb{C}^{n+2}_\Phi$ and $\mathbb{C}^{n+2}_\Theta$ whose set $\mathcal{J}(\mathbb{C}^{n+2}_\Phi)$ and $\mathcal{J}(\mathbb{C}^{n+2}_\Theta)$ are respectively

$$
\mathcal{J}(\mathbb{C}^{n+2}_\Phi) = \left\{\mu_0 = 0,\;\; J_1 = 2 \;\;\; J^{\mu_0}_{k+1}(\underbrace{0,\ldots,0}_{k-times}) = J^{\mu_0}_{k+1}(\underbrace{1,\ldots,0}_{k-times}) = 1,\;\; 1\leq k\leq n\right\}
$$

$$
\mathcal{J}(\mathbb{C}^{n+2}_\Theta) = \left\{\mu_0 = \omega_n,\;\; J^{\mu_0}_1 = 1,\;\; J^{\mu_0}_{k+1}(\underbrace{0,\ldots,0}_{k-times}) = 1,\;\; 1\leq k< n,\; J^{\mu_0}_{n+1}(\underbrace{0,\ldots,0}_{n-times}) = 2\right\}.
$$

The aim of this section is to investigate the restrictions through such embeddings of the finite dimensional irreducible $\mathfrak{sl}(n+2)$–modules to $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$.

Observe that the automorphism Ξ defined in Proposition 2.7 can be extended to the whole weight space $\mathfrak{h}^*$ by setting

$$\Xi(\lambda) = \Xi(\sum_{k=1}^{n+1} \lambda_k \omega_k) = \sum_{k=1}^{n+1} \lambda_{n+2-k} \omega_k, \qquad \lambda \in \mathfrak{h}^*$$

and that therefore it holds

Proposition 4.1 *Let $V(\lambda)$ be a irreducible finite dimensional $\mathfrak{sl}(n+2)$–module, then $V(\lambda)$ viewed as $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module using the embedding Φ (2.6) is equivalent to the module $V(\Xi(\lambda))$ always viewed as $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module using the embedding Θ (2.6)*

Therefore, since the automorphism Ξ preserves the integral dominant weight of $\mathfrak{sl}(n+2)$, in order to obtain all the modules of $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$, given by the restriction to it of the irreducible finite dimensional $\mathfrak{sl}(n+2)$–modules it is enough to consider one of the two embedding, for our convenience let us chose Φ (2.6).

Although this fact would allow us to simply talk about restriction of $\mathfrak{sl}(n+2)$–modules to $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ in order to avoid confusion we shall denote by $V(\lambda)_\Phi$ the restriction to $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ of the irreducible $\mathfrak{sl}(n+2)$–module $V(\lambda)$.

Observe that if u_μ is a weight vector of $\mathfrak{sl}(n+2)$ on $V(\lambda)$ then it is a weight vector of $\mathfrak{sl}(n+1)$ in $V(\lambda)_\Phi$ of weight $\mu = \sum_{k=2}^{n+1} \mu_k \omega_k$. In particular if u_μ is a highest weight vector of $\mathfrak{sl}(n+1)$ in $V(\lambda)_\Phi$ then using the embedding (2.6) the element $\varphi_i(u_\mu)$, $1 \le i \le n+1$ (3.11) are

$$\varphi_i(u_\mu) = F_{1,i}^{((\mu(i))}(u_\mu) + \sum_{k=1}^{i-1} Q_{k+1}^{(\mu(i))} F_{1,k}^{((\mu(i))}(u_\mu) \qquad 1 \le i \le n+1 \tag{4.1}$$

where the elements $Q_{k+1}^{(\mu(i))}$, $1 \le k \le n = 1$ are those given by Definition 3.10 applied to the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(n+2))$. For instance the first elements $\varphi_i(u_\mu)$ become

$$\begin{aligned}
\varphi_1(u_\mu) &= F_1 u_\mu \\
\varphi_2(u_\mu) &= -(\mu_2+1)F_{1,2}u_\mu + F_2F_1u_\mu \\
\varphi_3(u_\mu) &= (\mu_3+1)(\mu_3+\mu_2+2)F_{1,3}u_\mu - (\mu_3+\mu_2+2)F_3F_{1,2}u_\mu \\
&\quad -(\mu_3+1)F_{2,3}F_1u_\mu + F_3F_2F_1u_\mu \\
\varphi_4(u_\mu) &= -(\mu_4+1)(\mu_4+\mu_3+2)(\mu_4+\mu_3+\mu_2+3)F_{1,4}u_\mu \\
&\quad +(\mu_4+\mu_3+2)(\mu_4+\mu_3+\mu_2+3)F_4F_{1,3}u_\mu \\
&\quad +(\mu_4+1)(\mu_4+\mu_3+\mu_2+3)F_{3,4}F_{1,2}u_\mu - (\mu_4+\mu_3+\mu_2+3)F_4F_3F_{1,2}u_\mu \\
&\quad +(\mu_4+1)(\mu_4+\mu_3+2)F_{2,4}F_1u_\mu - (\mu_4+\mu_3+2)F_4F_{2,3}F_1u_\mu \\
&\quad -(\mu_4+1)F_{3,4}F_2F_1u_\mu + F_4F_3F_2F_1u_\mu.
\end{aligned}$$

Theorem 4.2 *For any integral dominant weight $\lambda = \sum_{i=1}^{n+1} \lambda_i\omega_i$ of $\mathfrak{sl}(n+2)$ the restriction $V(\lambda)_\Phi$ of the irreducible $\mathfrak{sl}(n+2)$–module $V(\lambda)$ to $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ is a cyclic module, with generator v_λ the $\mathfrak{sl}(n+2)$–highest weight vector in $V(\lambda)$, whose $\mathfrak{sl}(n+1)$–weight is $\lambda_0 = \sum_{k=2}^{n+1} \lambda_k\omega_k$.*

Proof Define on the set Δ^+ of positive roots of $\mathfrak{sl}(n+2)$ a total order $\succ_n$:

$$\alpha_{p,q} \succ_n \alpha_{r,s} \qquad \Longleftrightarrow \qquad \text{if} \quad p < r \quad \text{or} \quad p = r \ \text{ and } \ q < s \tag{4.2}$$

Correspondingly, we say that $F_{p,q} \succ_n F_{r,s}$ if $\alpha_{p,q} \succ_n \alpha_{r,s}$ so

$$F_{1,1} \succ_n \cdots F_{1,n+1} \succ_n F_{2,2} \succ_n F_{2,3} \succ_n \cdots \succ F_{n-1,n+1} \succ_n F_{n,n} \succ_n F_{n,n+1} \succ_n F_{n+1,n+1}.$$

Then using this ordering the $\mathfrak{sl}(n+2)$–module $V(\lambda)$ is the span of vectors of the type [13] [11]

$$F_{n+1}^{a_{n+1}} F_{n,n+1}^{a_{n,n+1}} \cdots F_{i,j}^{a_{i,j}} \cdots F_{1,n}^{a_{1,n}} F_{1,n-1}^{a_{1,n-1}} \cdots F_1^{a_1} v_\lambda \quad a_{i,j} \geq 0.$$

Applying the embedding Φ (2.6) this means that

$$V(\lambda) = \mathcal{U}(\mathfrak{sl}(n+1))\mathcal{U}(\mathfrak{p})v_\lambda$$

i.e., that $V(\lambda)$ is a cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module with generators v_λ.

$\square$

Theorem 4.3 *Let $V(\lambda)$ be an irreducible finite dimensional $\mathfrak{sl}(n+2)$–module of highest weight $\lambda = \sum_{k=1}^{n+1} \lambda_k\omega_k$. Then for the corresponding cyclic module $V(\lambda))_\Phi$ of $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ we have*

$$\mathcal{J}(V(\lambda)_\Phi)) = \left\{ \begin{array}{l|l} \lambda_0 = \sum_{i=2}^{n+1} \lambda_i\omega_i, \ \ J_1^{\lambda_0} = \lambda_1 + 1, & 0 \leq i_h \leq \lambda_h, \ 1 \leq h \leq n+1 \\ J_k^{\lambda_0}(i_1, \cdots i_{k-1}) = \lambda_k + 1 & 2 \leq k \leq n+1 \end{array} \right\}. \tag{4.3}$$

Proof Let v_{λ_0} be the highest weight vector of $\mathfrak{sl}(n+2)$–module $V(\lambda)$ seen as highest weight vector of $\mathfrak{sl}(n+1)$ with weight λ_0.

Since from equations (2.6) and (3.11) we have

$$\varphi_1^k(v_{\lambda_0}) = F_1^k(v_{\lambda_0}),$$

formula $J_1^{\lambda_0} = \lambda_1 + 1$ follows from the theory of finite dimensional $\mathfrak{sl}(2)$–modules.

We can now proceed by induction suppose that we have already shown $J_h^{\lambda_0}(i_1, \cdots i_{h-1}) = \lambda_h + 1$ for $h \leq k$, we want show that $J_{k+1}^{\lambda_0}(i_1, \cdots, i_k) = \lambda_{k+1} + 1$. First we shall show that

$$\varphi_{k+1}^{i_{k+1}}(\varphi_k^{i_k}(\cdots(\varphi_1^{i_1}(v_{\lambda_0}))) \neq 0 \quad \text{if } i_{k+1} \leq \lambda_{k+1}, 0 \leq i_h \leq \lambda_h, 1 \leq h \leq k.$$

Let u_η be defined by

$$u_\eta = \varphi_k^{i_k}(\cdots(\varphi_1^{i_1}(v_{\lambda_0})).$$

We want show that if $\lambda_{k+1} \geq 1$ then

$$\varphi_{k+1}(u_\eta) \neq 0.$$

From Theorem 3.13 the vector u_η is a highest weight vector of $\mathfrak{sl}(n+1)$ of $\mathfrak{sl}(n+2)$ weight

$$\eta = \sum_{j=1}^{n+1} \eta_k \omega_k = \sum_{j=1}^{k-1} (\lambda_j + i_j - i_{j+1})\omega_j + (\lambda_k + i_k)\omega_k + \sum_{j=k+1}^{n+1} \lambda_j \omega_j, \tag{4.4}$$

and therefore of $\mathfrak{sl}(n+1)$ weight $\eta^{(2)} = \sum_{j=2}^{n+1} \eta_k \omega_k$. Hence using equations (2.6) and (4.1) we have

$$\varphi_{k+1}(u_\eta) = F_{1,k+1}^{(\eta(k+1))} u_\eta + \sum_{j=1}^{k} Q_{j+1}^{(\eta(k+1))} F_{1,j}^{(\eta(k+1))} u_\eta.$$

Since $\eta_{k+1} = \lambda_{k+1} \geq 1$, from the theory of the finite dimensional $\mathfrak{sl}(2)$–module (and from Theorem 2.2 as well) follows that $F_{1,k+1}^{(\eta(k+1))} u_\eta \neq 0$. Hence if $Q_{j+1}^{(\eta(k+1))} F_{1,j}^{(\eta(k+1))} u_\eta = 0$ for any $1 \leq j \leq k$ we have

$$\varphi_{k+1}(u_\eta) = F_{1,k+1}^{(\eta(k+1))} u_\eta \neq 0.$$

We may suppose, therefore, without loosing generality, that $Q_{j+1}^{(\eta(k+1))} F_{1,j}^{(\eta(k+1))} u_\eta \neq 0$ for $1 \leq j \leq k$. From Proposition 3.10 and formula (2.6) we have that

$$Q_{j+1}^{(\eta(k+1))} = R_{j+1,k+1-j}^{(\eta(k+1))} = \sum_{l=0}^{k-j} F_{j+1+l,k+1}^{(\mu(k+1))} R_{j+1,l}^{(\eta(k+1))} \tag{4.5}$$

and therefore

$$Q_{j+1}^{(\eta(k+1))} F_{1,j}^{(\eta(k+1))} u_\eta = \sum_{l=0}^{k-j} F_{j+l+1,k+1}^{(\mu(k+1))} R_{j+1,l}^{(\eta(k+1))} F_{1,j}^{(\eta(k+1))} u_\eta. \tag{4.6}$$

Consider now the simple subalgebra of $\mathfrak{sl}(n+2)$ generated by the elements $E_{p,q}$,$F_{p,q}$, with $1 \leq p \leq q \leq k$, which is isomorphic to $\mathfrak{sl}(k+1)$. The vector u_η is also a highest weight vector of such simple subalgebra with highest weight vector $\eta^{(k)} = \sum_{l=1}^{k} \eta_l \omega_l$.

Now form Definition 3.9 we have that the vectors $R_{j+1,l}^{(\eta(k+1))} F_{1,j}^{(\eta(k+1))} u_\eta$ belong to the the $\mathfrak{sl}(k+1)$-module generated by u_η. Hence we can write them as linear combination of the element of the (FFL) basis 2.2 of the irreducible finite dimensional $\mathfrak{sl}(k+1)$–module $V(\eta_k)$ of highest weight η_k and highest weight vector u_η:

$$R_{j+1,l}^{(\eta(k))} F_{1,j}^{(\eta(k+1))} u_\eta = \sum_{\mathbf{s}_k \in S(\eta_k)} c_{\mathbf{s}_k} F^{\mathbf{s}_k} u_\eta$$

where $S(\eta_k)$ is the set defined in Theorem 2.2. Substituting this last equation in (4.6) yields

$$Q_{j+1}^{(\eta(k+1))} F_{1,j}^{(\eta(k+1))} u_\eta = \sum_{l=0}^{k-j} \sum_{\mathbf{s}_k \in S(\eta_k)} F_{j+l+1,k+1}^{(\eta(k+1))} F^{\mathbf{s}_k} u_\eta = \sum_{l=j+1}^{k+1} \sum_{\mathbf{s}_k \in S(\eta_k)} F_{l,k+1}^{(\eta(k+1))} F^{\mathbf{s}_k} u_\eta.$$

We claim that if $F^{\mathbf{s}_k}u_\eta$ is an element of the Feigin Fourier Littelmann basis of the irreducible finite dimensional $\mathfrak{sl}(k+1)$–module $V(\eta_k)$ then $F_{l,k+1}^{\eta(k+1)}F^{\mathbf{s}_k}u_\eta$ is element of the Feigin Fourier Littelmann basis of the $\mathfrak{sl}(n+2)$–module $V(\eta)$ of highest weight η. Let $\mathbf{s} = (\mathbf{s}_k, \mathbf{s}_{l,k+1})$ the multi–exponent of $\mathfrak{sl}(n+2)$ such that $F^{\mathbf{s}}u_\eta = F_{l,k+1}^{(\eta(k+1))}F^{\mathbf{s}_k}u_\eta$ for Theorem 2.2 we have to show that for any Dyck path $\mathbf{p} = (\beta(0), \ldots, \beta(h))$ with say $\beta(0) = \alpha_r$ and $\beta(h) = \alpha_s$ it holds

$$s_{\beta(0)} + s_{\beta(1)} + \cdots + s_{\beta(h)} \leq \sum_{i=r}^{s} \eta_i. \tag{4.7}$$

Let Δ_i^+, $1 \leq i \leq n+1$ be the subset of the set of the positive roots Δ^+ of $\mathfrak{sl}(n+2)$ which can be written as linear combination of the simple roots α_l with $1 \leq l \leq i$ (i,e, the set of the positive roots of the subalgebra $\mathfrak{sl}(i+1)$). Then for any Dyck path $\mathbf{p} = (\beta(0), \ldots, \beta(h))$ of $\mathfrak{sl}(n+2)$ we have

$$(s_k)_{\beta(i)} = 0 \text{ if } \beta(i) \notin \Delta_k^+ \quad \text{and} \quad (s_{l,k+1})_{\beta(i)} = 0 \text{ if } \beta(i) \notin \Delta_{k+1}^+. \tag{4.8}$$

Now if the elements of the Dyck path $\mathbf{p}$ belong to Δ_k^+ i.e, if $\beta(h) = \alpha_s$ with $1 \leq s \leq k$ equation (2.2) is verified because $\mathbf{s}_k \in S(\eta_k)$. While, if this is not the case, using equation (4.8), we may assume that $\beta(h) = \alpha_{k+1}$, then equation (4.7) becomes $\sum_{i=r}^{k+1} s_{\beta(i)} \leq \sum_{i=r}^{k+1} \eta_i$. Since $\eta_{k+1} \geq 1$ if $\beta(0) = \beta(h) = \alpha_{k+1}$ equation (4.7) is obviously verified since $(s_{l,k+1})_{\alpha_{k+1}} \leq 1$ otherwise let be $(\beta(0), \ldots, \beta(l))$ the subset given by the root of $\mathbf{p}$ which belong to Δ_k^+ then from the very definition 2.1 of Dyck path it follows that there exist a p with $1 \leq p \leq k$ such that $\beta(l) = \alpha_{p,k}$. Then adding to the end of the sequence $(\beta(0), \ldots, \beta(l))$ the elements $(\alpha_{p+1,k}, \alpha_{p+2,k}, \ldots, \alpha_{k,k})$, we make it a Dyck path of $\mathfrak{sl}(k+1)$. Hence we have that equation (4.7) is satisfied because

$$\sum_{j=0}^{l} s_{\beta(j)} + \sum_{j=l+1}^{h} s_{\beta(j)} \leq \sum_{j=0}^{l}(s_k)_{\beta(j)} + (s_{1,k+1})_{\alpha_{l,k+1}} \leq \sum_{j=0}^{l}(s_k)_{\beta(j)} + \sum_{t=1}^{k-p}(s_k)_{\alpha_{p+t,k}} + 1 \leq \sum_{i=r}^{k} \eta_k + \eta_{k+1}$$

being $\mathbf{s}_k \in S(\eta_k)$ and $\eta_{k+1} \geq 1$.

Now $\eta_{k+1} > 1$ and Theorem 2.2 imply also that $F_{1,k+1}^{(\eta(k+1))}u_\eta$ is an element of the (FFL) basis of the $\mathfrak{sl}(n+2)$–module $V(\eta)$.

Therefore $\varphi_{k+1}(u_\eta)$ can be write as linear combination of elements such basis as

$$\varphi_{k+1}(u_\eta) = F_{1,k+1}^{(\eta(k+1))}u_\eta + \sum_{j=1}^{k}\sum_{l=j+1}^{k+1}\sum_{\mathbf{s}_k \in S(\eta_k)} F_{l,k+1}^{\eta(k+1)}F^{\mathbf{s}_k}u_\eta. \tag{4.9}$$

Finally, since the element $F_{1,k+1}^{(\mu(k+1))}u_\eta$ of the (FFL) basis appears only once in (4.9) and it is different from zero we have $\varphi_{k+1}(u_\eta) \neq 0$.

Summarizing we have shown that

$$\lambda_{k+1} \geq 1 \Longrightarrow F_{1,k+1}^{((\eta(k+1))}(u_\eta) \neq 0 \Longrightarrow \varphi_{k+1}(u_\eta) \neq 0.$$

Set now $u_\eta^l = \varphi_{k+1}^l(u_\eta)$, $0 \leq l \leq \lambda_{k+1}$ $(u_\eta^0 = u_\eta)$ the same argument shows that

$$\lambda_{k+1} \geq l \Longrightarrow F_{1,k+1}^{((\eta(k+1))}(u_\eta^{l-1}) \neq 0 \Longrightarrow \varphi_{k+1}(u_\eta^{l-1}) \neq 0.$$

Therefore for any $i_h \leq \lambda_h$, $1 \leq h \leq k$ we have $J^{\lambda_0}_{k+1}(i_1, \ldots, i_k) \geq \lambda_{k+1} + 1$.

For the Theorem 3.25 (3), to prove that $J^{\lambda_0}_{k+1}(i_1, \ldots, i_k) \leq \lambda_{k+1} + 1$ we need only to show that

$$\varphi^{\lambda_{k+1}+1}(v_{\lambda_0}) = 0.$$

But arguing like in the proof of Theorem 3.25 we see that if $\varphi^{\lambda_k+1}(v_{\lambda_0})$ is different from zero then there would be a highest weight vector of $\mathfrak{sl}(n+1)$ with a non dominant weight, which is impossible.

$\square$

Corollary 4.4 *The cyclic $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module $V(\lambda)_\Phi$ decomposes as $\mathfrak{sl}(n+1)$–module as*

$$V(\lambda)_\Phi = \bigoplus_{l=1}^{n+1} \bigoplus_{k_l=1}^{\lambda_l} V\left(\lambda_0 + \sum_{l=1}^{n}(k_l - k_{l+1})\omega_l\right).$$

Definition 4.5 *Let $\mathfrak{M}$ be the collection of all sets $\mathcal{M} = (\mu_0, \mathcal{M}^{\mu_0})$ such that $\mu_0 = \sum_{l=1}^{n} \mu_k \omega_k$ is a dominant weight of $\mathfrak{sl}(n+1)$ and $\mathcal{M}^{\mu_0}$ is a set of positive integers defined as follows:*

$$\mathcal{M}^{\mu_0} = \left\{ \begin{array}{l|l} & 0 \leq i_k < M^{\mu_0}_k(i_1, \ldots, i_{k-1}), \ \ 1 \leq k \leq n+1; \\ & M^{\mu_0}_1 \geq 1; \\ M^{\mu_0}_1, M^{\mu_0}_k(i_1, \ldots i_{k-1}), & 1 \leq M^{\mu_0}_k(i_1, \ldots i_{k-1}) \leq \mu_{k-1} + 1, \ \ 2 \leq k \leq n+1; \\ 2 \leq k \leq n+1 & M^{\mu_0}_k(i_1, \ldots, i_{k-1}) \geq M^{\mu_0}_k(j_1, \ldots, j_{k-1}) \\ & \textit{if } \ i_l \leq j_l, \ \ 0 \leq l \leq k-1 \ \ 2 \leq k \leq n+1. \end{array} \right\} \tag{4.10}$$

Theorem 4.6 *For any set $\mathcal{M} \in \mathfrak{M}$ there exist a cyclic module V of $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ such that*

$$\mathcal{M} = \mathcal{J}(V).$$

Such $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module V can be constructed as quotient of a suitable $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–module $V(\lambda)_\Phi$ obtained as restriction to $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$ of a finite dimensional irreducible $\mathfrak{sl}(n+2)$–module $V(\lambda)$.

Proof Let $\mathcal{M}$ be a set in $\mathfrak{M}$ (see Definition 4.6) and let $\lambda_0 = \sum_{k=1}^{n} \lambda^0_k \omega_k$ be the integral dominant weight in $\mathcal{M}$.

Define the dominant integral weight λ of $\mathfrak{sl}(n+2)$ as follows

$$\lambda = \sum_{k=1}^{n+1} \lambda_k \omega_k \quad \lambda_1 = M^{\lambda_0}_1 - 1, \ \lambda_{k+1} = \lambda^0_{k-1} \ \ 2 \leq k \leq n$$

where $M^{\lambda_0}_1$ is the positive integer which appears in the definition of $\mathcal{M}$. Let $V(\lambda)$ be the associated irreducible finite dimensional $\mathfrak{sl}(n+2)$–module with highest weight vector v_λ,

and let $V(\lambda)_\Phi$ be its restriction to $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$. From equation (4.3) and the Definition 4.5 follows that

$$M_1^{\lambda_0} = \lambda_1 + 1$$
$$M_k^{\lambda_0}(i_1, \ldots, i_{k-1}) \leq \lambda_k + 1 \quad 0 \leq i_h < M_h^{\mu_0}(i_1, \ldots, i_{h-1}),\ 1 \leq h \leq k-1 \quad 2 \leq k \leq n+1$$

For any integer k with $1 \leq k \leq n$, let $\mathcal{I}_k = \{i_1, \ldots, i_k\}$ be the set of positive integer numbers such that $i_l < M_l^{\lambda_0}(i_1, \ldots, i_{l-1})$, $1 \leq l \leq k$ and $M_{k+1}^{\lambda_0}(i_1, \ldots, i_k)$, is strictly less then $\lambda_k + 1$. Further for any $1 \leq k \leq n+1$ and any k-uple $\{i_1, \ldots, i_k\} \in \mathcal{I}_k$ let $u^{(k)}(i_1, \ldots .i_k)$ be defined by

$$u^{(k)}(i_1, \ldots, i_k) = \varphi^{M_{k+1}^{\lambda_0}(i_1, \ldots, i_{k-1})+1}(\varphi_{i_{k-1}}^{i_{k-1}}(\cdots(\varphi_1^{i_1}(v_\lambda)))).$$

Finally for any $1 \leq k \leq n+1$ and any k-uple $\{i_1, \ldots, i_k\} \in \mathcal{I}_k$ denote by $W^{(k)}(i_1, \ldots, i_k)$ the $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–submodule generated by $u^{(k)}(i_1, \ldots, i_k)$:

$$W^{(k)}(i_1, \ldots, i_k) = \mathcal{U}(\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}) u^{(k)}(i_1, \ldots i_k)$$

and with $W_{\mathcal{M}}$ their union

$$W_{\mathcal{M}} = \bigcup_{k=1}^{n} \left(\bigcup_{(i_1, \ldots, i_k) \in \mathcal{I}_k} W^{(k)}(i_1, \ldots, i_k) \right).$$

Let $V_{\mathcal{M}}$ be finally the quotient module

$$V_{\mathcal{M}} = (V(\lambda))_\Phi / W_{\mathcal{M}}.$$

Then by construction from Theorems 3.25, 4.3 and the Definition 4.5 it follows

$$\mathcal{J}(V_{\mathcal{M}}) = \mathcal{M}.$$

□

Putting together Theorem 3.27 and Theorem 4.6 we have

Theorem 4.7 *The class of the finite dimensional $\mathfrak{sl}(n+1) \ltimes \mathbb{C}^{n+1}$–modules is in one to one corresponence with the collection of sets $\mathfrak{M}$ defined in Definition 4.5.*

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